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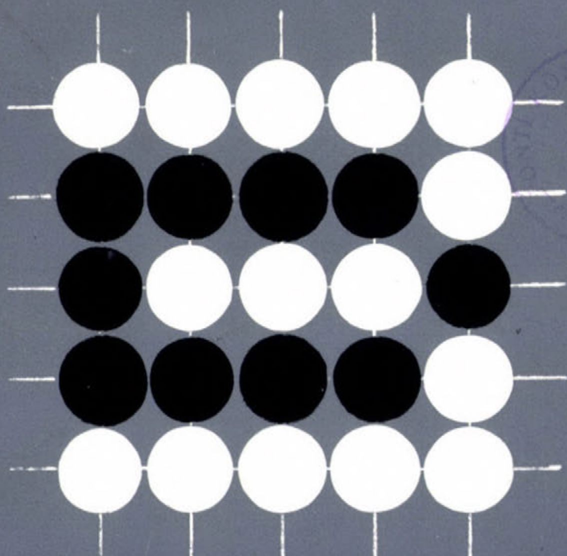

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SZÁMÍTÁSTECHNIKAI ÉS AUTOMATIZÁLÁSI KUTATÓ INTÉZET

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Szepesvári István:

KONVERGENS VÉGES DIFFERENCIA MÓDSZER BIZONYOS DEGENERÁLT NEMLINEÁRIS TÖBBVÁLTOZÓS PARABOLIKUS EGYENLETRE

1. BEVEZETÉS

J.L. Graveleau és P. Jamet az [1] dolgozatban a

$$\frac{\partial u}{\partial t} = f(x,t,u)\frac{\partial^2 u}{\partial x^2} + a\left(\frac{\partial u}{\partial x}\right)^2$$

egyenlet numerikus megoldásával foglalkozik az $u(x,0) = u^0(x)$ kezdeti feltétel mellett, ha $f(x,t,u) \geqslant 0$, $a \geqslant 0$ konstans, x egydimenziós. Ebben a cikkben – továbbfejlesztve az [1]-ben alkalmazott módszereket – az előbbinél általánosabb (1.1) – (1.2) egyenletre adunk stabilis és konvergens véges differencia közelítést.

Tekintsük a következő differenciálegyenletet:

$$(1.1) \qquad \frac{\partial u(x,t)}{\partial t} = \sum_{m=1}^{s} f_m(x,t,u(x,t),u_{x_m}(x,t))\frac{\partial^2 u(x,t)}{\partial x_m^2} + \sum_{p=1}^{r}\sum_{m=1}^{s} c_{m,p}(t)\left(\frac{\partial u(x,t)}{\partial x_m}\right)^p \equiv Au$$

az

$$(1.2) \qquad u(x,0) = u^0(x)$$

kezdeti feltétellel, ahol $x = (x_1, \ldots, x_s) \in \mathscr{R}^s$, $p \geqslant 1$ egész.

Feltesszük, hogy $f_m(x,t,u,u_{x_m})$ folytonos, nemnegatív, valós változós függvény;

$\dfrac{\partial f}{\partial x_m}$, $\dfrac{\partial f}{\partial u}$, $\dfrac{\partial}{\partial x_m}\left(\dfrac{\partial^n f}{\partial u_{x_m}^n}\right)$, $\dfrac{\partial}{\partial u}\left(\dfrac{\partial^n f}{\partial u_m^n}\right)$ folytonos $m = 1, \ldots, s$-re és $1 \leqslant n \leqslant N$;

$\dfrac{\partial^n}{\partial u_{x_m}^n} f \equiv 0$ $\quad n > N$-re; $c_{m,p}(t)$ nemnegatív, $\sup\limits_t |c_{m,p}(t)|$, $\sup\limits_x |u^0(x)|$ és $\sup\limits_x \left|\dfrac{\partial u^0}{\partial x_m}\right|$

korlátos, $\dfrac{\partial u^0}{\partial x_m} \in \mathscr{L}^t(\mathscr{R}^s)$, $\operatorname*{Var}\limits_{x_m}\left(\dfrac{\partial u^0}{\partial x_m}\right) \in \mathscr{L}^1(\overline{\mathscr{R}}_m)$, ahol $\operatorname*{Var}\limits_{x_m}$ az x_m szerinti variáció,

$\overline{\mathscr{R}}_m = (x_1, \ldots, x_{m-1}, x_{m+1}, \ldots, x_s)$; $\quad m = 1, \ldots, s$, $\quad t = \max(r, N+2)$.

Definíció. Legyen $\mathscr{H}\{x \in \mathscr{R}^s,\ t > 0\}$ féltér. Egy $\mathscr{H}$-n definiált $u(x,t)$ függvény az (1.1) – (1.2) probléma megoldása, ha

$$u \in C^0(\overline{\mathscr{H}}) \cap \mathscr{L}^\infty(\mathscr{H}),\ u_{x_m} \in \mathscr{L}^\infty(\mathscr{H}),\ m = 1, \ldots, s;\ u(x,0) = u^0(x)$$

és

Beérkezett: 1973. december 12.

$$
(1.3) \quad -\int_{\mathcal{H}}\Bigg[u\frac{\partial\Phi}{\partial t}+\sum_{m=1}^{s}\Bigg\{f_m u_{x_m}\Phi_{x_m}+\left(\frac{\partial f_m}{\partial x_m}u_{x_m}+\frac{\partial f_m}{\partial u}(u_{x_m})^2\right)\Phi+
+\sum_{k=1}^{N}\left(\frac{(-1)^{k+1}}{(k+1)!}\left[\frac{\partial^k f_m}{\partial u^k_{x_m}}(u_{x_m})^{k+1}\Phi_{x_m}+\left(\frac{\partial}{\partial x_m}\left(\frac{\partial^k f_m}{\partial u^k_{x_m}}\right)\right)(u_{x_m})^{k+1}\Phi+\right.\right.
\left.\left.+\left(\frac{\partial}{\partial u}\left(\frac{\partial^k f_m}{\partial u^k_{x_m}}\right)\right)(u_{x_m})^{k+2}\Phi\right]\right)-\left(\sum_{p=1}^{r}\sum_{m=1}^{s}c_{m,p}(t)\left(\frac{\partial u(x,t)}{\partial x_m}\right)^p\right)\Phi\Bigg\}\Bigg]=0
$$

minden $\Phi\in\mathcal{D}(\mathcal{H})$ esetén, ahol $\mathcal{D}(\mathcal{H})$ a $\mathcal{H}$-ban kompakt tartójú végtelenszer differenciálható függvények terét jelenti, $\overline{\mathcal{H}}$ pedig a $\mathcal{H}$ lezártja.

Az (1.3) összeget a következőképpen kapjuk:

$$
f(x,t,u,u_x)u_{xx}=\frac{\partial}{\partial x}(fu_x)-f_x u_x-f_u u_x^2-f_{u_x}u_{xx}u_x.
$$

Továbbá:

$$
(1.4) \quad f_{u_x}u_{xx}u_x=\frac{1}{2}\frac{\partial}{\partial x}(f_{u_x}u_x^2)-\frac{1}{2}(f_{u_x x}u_x^2+f_{u_x u}u_x^3+f_{u_x u_x}u_x^2 u_{xx}).
$$

(1.4) utolsó tagjára ismét alkalmazva egy (1.4)-hez hasonló képletet, figyelembe véve, hogy $j(u_x)^{j-1}u_{xx}=\frac{\partial}{\partial u}(u_x)^j$ ($j>1$, egész), majd hasonló módon folytatva, amíg $\frac{\partial^n}{\partial u_x^n}f\equiv 0$ lesz, parciális integrálás után kapjuk (1.3)-at.

Az (1.1) egyenlet speciális esetei:

$$
(1.41) \quad \frac{\partial u}{\partial t}=mu\frac{\partial^2 u}{\partial x^2}+\frac{m}{m-1}\left(\frac{\partial u}{\partial x}\right)^2, \qquad m>1
$$

és $m=2$-re

$$
(1.42) \quad \frac{\partial u}{\partial t}=\frac{\partial^2}{\partial x^2}(u^2) \qquad \text{(Boussinesq egyenlet)}
$$

Bevezetve az $u=v^{m-1}$ transzformációt, (1.41) a következőképpen alakul:

$$
(1.43) \quad \frac{\partial v}{\partial t}=\frac{\partial^2}{\partial x^2}(v^m)
$$

amely egyrészt a lukacsos közegen való átfolyást, másrészt a sugárzásos hőátadás jelenségét írja le.

Cikkünkben egy explicit véges differencia sémát állítunk fel az (1.1)-(1.2) feladat megoldására, majd lokális egyenletes konvergenciát bizonyítunk. Differencia sémánk az A operátor feldarabolásán alapul: $Au=Bu+Cu$,

$$(1.5) \qquad Bu \equiv \sum_{m=1}^{s} f_m \frac{\partial^2 u}{\partial x_m^2}$$

$$(1.6) \qquad Cu \equiv \sum_{p=1}^{r} C_p u\,,$$

ahol

$$(1.7) \qquad C_p u = \sum_{m=1}^{s} c_{m,p}(t) \left(\frac{\partial u(x,t)}{\partial x_m} \right)^p \qquad p = 1, \dots, r.$$

2. NUMERIKUS SÉMA (1.5)-RE. STABILITÁSI BECSLÉSEK

Legyenek $h_1, \dots, h_s, k$ pozitív számok. Tekintsük a következő $s+1$ dimenziós rácsponttartományt: $H \equiv \mathscr{H}_{h_1,\dots,h_s,k} = \{(x,t) = (i_1 h_1, \dots, i_s h_s, nk);\ i_j,\ j = 1, \dots, s$ és n egész, $n \geqslant 0\}$.

Legyen Ψ^n tetszőleges H-n definiált függvény, $\Psi^n \equiv \Psi^n_{i_1,\dots,i_s}(i_1 h_1, \dots, i_s h_s, nk)$, valamint:

$$(2.1) \qquad B_{h_m} \Psi^n \equiv \sum_{m=1}^{s} B_{m,h_m} \Psi^n\,,$$

ahol

$$B_{m,h_m} \Psi^n = f_{m,i_m}\left(i_1 h_1, \dots, i_s h_s, nk, \Psi^n, \frac{\Psi^n_{i_1,\dots,i_m+1,\dots,i_s} - \Psi^n_{i_1,\dots,i_m,\dots,i_s}}{h_m} \right) *$$

$$* \frac{\Psi^n_{i_1,\dots,i_m+1,\dots,i_s} - 2\Psi^n + \Psi^n_{i_1,\dots,i_m-1,\dots,i_s}}{h_m^2}$$

Tekintsük a következő véges differencia sémát:

$$(2.2) \qquad (u^{n+1}_{i_1,\dots,i_s} - u^n_{i_1,\dots,i_s}) / k = \sum_{m=1}^{s} B_{m,h_m} u^n_{i_1,\dots,i_s}$$

$$u^0_{i_1,\dots,i_s} = u^0(i_1 h_1, \dots, i_s h_s)$$

amely megfelel az (1.5)-(1.2) feladatnak.

2.1. Tétel. *Legyen*

$$\lambda_m = \frac{k}{h_m^2}\,, \qquad C_0 = \sup_x |u^0(x)|\,,$$

$$C_{1,m} = \sup_x \left| \frac{\partial u^0(x,t)}{\partial x_m} \right| , \qquad M_m = \sup_{\substack{|p| \leq C_0 \\ |q| \leq C_{1,m} \\ (x,t) \in \mathscr{H}}} |f_m(x,t,p,q)| .$$

Tegyük fel, hogy

$$(2.3) \qquad 2M_m \lambda_m \bar{s} \leq 1 , \qquad m = 1, \dots , s,$$

ahol $\bar{s}$ a nem azonosan 0 f_m-ek száma.

Ekkor a következő becslések állnak:

$$(2.4) \qquad |u^n_{i_1, \dots, i_s}| \leq C_0 \qquad \text{minden} \quad i_1, \dots , i_s, n\text{-re},$$

$$(2.5) \qquad |(u^n_{i_1, \dots, i_m + 1, \dots, i_s} - u^n_{i_1, \dots, i_m, \dots, i_s}) / h_m | \leq C_{1,m} \qquad \text{minden} \quad i_1, \dots , i_s, n\text{-re},$$

$$(2.6) \qquad h_1 \dots h_s \sum_{i_1, \dots, i_s} |(u^n_{i_1, \dots, i_m + 1, \dots, i_s} - u^n_{i_1, \dots, i_m, \dots, i_s}) / h_m |^p \leq C_{2,m,p}$$

$$\text{minden} \quad n\text{-re},$$

$$(2.7) \qquad h_1 h_2 \dots h_s \sum_{i_1, \dots, i_s} |(u^n_{i_1, \dots, i_m + 1, \dots, i_s} - 2u^n_{i_1, \dots, i_s} + u^n_{i_1, \dots, i_m - 1, \dots, i_s}) / h^2_m | \leq$$

$$\leq C_{3,m} \qquad \text{minden} \quad n\text{-re},$$

$$(2.8) \qquad h_1 h_2 \dots h_s \sum_{i_1, \dots, i_s} |(u^{n+1}_{i_1, \dots, i_s} - u^n_{i_1, \dots, i_s}) / k | \leq C_4 \qquad \text{minden} \quad n\text{-re},$$

ahol C_0; $C_{1,m}$; $C_{2,m,p}$; $C_{3,m}$; C_4 $h_1, \dots , h_s$ és k-tól független konstans,

$$M = \max_m M_m ; \quad C_3 = \max_m C_{3,m} ; \quad C_4 = \bar{s} M C_3 ;$$

$$C_{2,m,p} = \int_{\mathscr{R}^s} \left| \frac{\partial u^0}{\partial x_m} \right|^p ; \quad C_{3,m} = \int_{\bar{\mathscr{R}}_m} \left| \operatorname{Var}_{x_m} \left(\frac{\partial u^0}{\partial x_m} \right) \right| .$$

Bizonyítás. (2.2)-ből következik:

$$u^{n+1}_{i_1, \dots, i_s} = \sum_{m=1}^{\bar{s}} \left\{ \left(\frac{1}{\bar{s}} - 2\lambda_m f^n_{m,i_m} \right) u^n_{i_1, \dots, i_s} + \lambda_m f^n_{m,i_m} u^n_{i_1, \dots, i_m + 1, \dots, i_s} + \right.$$

$$\left. + \lambda_m f^n_{m,i_m} u^n_{i_1, \dots, i_m - 1, \dots, i_s} \right\} .$$

Mivel $0 \leq f^n_{m,i_m} \leq M_m$, így (2.3) miatt $0 \leq \lambda_m f^n_{m,i_m} \leq \dfrac{1}{2\bar{s}}$.

Ezért, minthogy $u^n_{i_1,\ldots,i_m-1,\ldots,i_s}$; $u^n_{i_1,\ldots,i_s}$; $u^n_{i_1,\ldots,i_m+1,\ldots,i_s}$ együtthatói nemnegatívak és összegük 1, tehát

$$|u^{n+1}_{i_1,\ldots,i_s}| \leqslant \max_{i_1,\ldots,i_s} \{|u^n_{i_1,\ldots,i_s}|, |u^n_{i_1,\ldots,i_m+1,\ldots,i_s}|, |u^n_{i_1,\ldots,i_m-1,\ldots,i_s}|\} .$$

Innen $\sup_{i_1,\ldots,i_s} |u^{n+1}_{i_1,\ldots,i_s}| \leqslant \sup_{i_1,\ldots,i_s} |u^n_{i_1,\ldots,i_s}|$, és (2.4) indukcióval következik.

Legyen $u^n_{i_1,\ldots,i_m+j,\ldots,i_s} \equiv u^n_{i_m+j}$ valamint $v^n_{i_m} = (u^n_{i_m+1} - u^n_{i_m})/h_m$. (2.2)-ből:

$$(v^{n+1}_{i_m} - v^n_{i_m})/k = \sum_{m=1}^{\bar{s}} (B_{m,h_m} u^n_{i_m+1} - B_{m,h_m} u^n_{i_m})/h_m ,$$

azaz

$$v^{n+1}_{i_m} = v^n_{i_m} + \sum_{m=1}^{\bar{s}} \lambda_m [f^n_{m,i_m+1}(v^n_{i_m+1} - v^n_{i_m}) - f^n_{m,i_m}(v^n_{i_m} - v^n_{i_m-1})] =$$

$$= \sum_{m=1}^{\bar{s}} \left[\frac{1}{\bar{s}} - \lambda_m (f^n_{m,i_m+1} - f^n_{m,i_m})\right] v^n_{i_m} + \lambda_m f^n_{m,i_m} v^n_{i_m+1} + \lambda_m f^n_{m,i_m} v^n_{i_m-1},$$

amiből az előzőhöz hasonlóan következik (2.5).

Másrészt

$$(2.81) \qquad |v^{n+1}_{i_m}| \leqslant \sum_{m=1}^{\bar{s}} \left[\frac{1}{\bar{s}} - \lambda_m (f^n_{m,i_m+1} + f^n_{m,i_m})\right] |v^n_{i_m}| + \lambda_m f^n_{m,i_m+1} |v^n_{i_m+1}| +$$

$$+ \lambda_m f^n_{m,i_m} |v^n_{i_m-1}| ,$$

és $|u^{n+1}_{i_m+1} - u^{n+1}_{i_m}| \leqslant |u^0_{i_m+1} - u^0_{i_m}|$ miatt a kezdeti feltétel szerinti egyenletes stabilitás következik.

Minden $i_1, \ldots, i_m$-re szummázva kapjuk (2.81)-ből:

$$\sum_{i_1,\ldots,i_s}' |v^{n+1}_{i_m}|^p \leqslant \sum_{i_1,\ldots,i_s} |v^n_{i_m}|^p$$

és indukcióval következik:

$$h_1 h_2 \ldots h_s \sum_{i_1,\ldots,i_s} |v^n_{i_m}|^p \leqslant h_1 h_2 \ldots h_s \sum_{i_1,\ldots,i_s} |v^0_{i_m}|^p \leqslant \bar{C}_{2,m,p};$$

ami korlátos.

Legyen most

$$w^n_{i_m} = (v^n_{i_m} - v^n_{i_m-1})/h_m = (u^n_{i_m+1} - 2u^n_{i_m} + u^n_{i_m-1})/h^2_m .$$

(2.2)-ből következik:

$$(w_{i_m}^{n+1} - w_{i_m}^{n})/k = \sum_{m=1}^{\bar{s}} (B_{m,h_m} u_{i_m+1}^{n} - 2B_{m,h_m} u_{i_m}^{n} + B_{m,h_m} u_{i_m-1}^{n})/h_m^2$$

(2.9)
$$w_{i_m}^{n+1} = w_{i_m}^{n} + \sum_{m=1}^{\bar{s}} \lambda_m (f_{m,i_m+1}^{n} w_{i_m+1}^{n} - 2f_{m,i_m}^{n} w_{i_m}^{n} + f_{m,i_m}^{n} w_{i_m-1}^{n}) =$$

$$= \sum_{m=1}^{\bar{s}} \left(\frac{1}{\bar{s}} - 2\lambda_m f_{m,i_m}^{n} \right) w_{i_m}^{n} + \lambda_m f_{m,i_m+1}^{n} w_{i_m+1}^{n} + \lambda_m f_{m,i_m-1}^{n} w_{i_m-1}^{n},$$

és összegzéssel kapjuk:

$$h_1 h_2 \ldots h_s \sum_{i_1,\ldots,i_s} |w_{i_m}^{n}| \leqslant h_1 h_2 \ldots h_s \sum_{i_1,\ldots,i_s} |w_{i_m}^{0}| \leqslant C_{3,m}\,.$$

A (2.8) becslés az előző becslésekből rögtön következik:

$$(u_{i_m}^{n+1} - u_{i_m}^{n})/k = \sum_{m=1}^{\bar{s}} f_{m,i_m}^{n} w_{i_m}^{n},$$

ahonnan:

$$h_1 h_2 \ldots h_s \sum_{m=1}^{\bar{s}} \sum_{i_1,\ldots,i_s} |(u_{i_m}^{n+1} - u_{i_m}^{n})/k| \leqslant \sum_{m=1}^{\bar{s}} h_1 h_2 \ldots h_s \sum_{i_1,\ldots,i_s} f_{m,i_m}^{n} |w_{i_m}^{n}| \leqslant$$

$$\leqslant M\bar{s}\, C_3\,.$$

3. AZ (1.6) VÉGES DIFFERENCIA ALAKBAN VALÓ ELŐÁLLITÁSA STABILITÁSI BECSLÉSEK

Tetszőleges H-n definiált Ψ függvényre legyen:

$$C\Psi^n \equiv \sum_{p=0}^{r-1} C_{p+1} \Psi^n,$$

ahol

$$C_{p+1}\Psi^n = \sum_{m=1}^{s} D_{m,p+1} \Psi_{i_m}^{n},$$

ahol

(3.1)
$$D_{m,p+1}\Psi_{i_m}^{n} = c_{m,p+1}(nk)(\delta\Psi_{i_m}^{n})^{p+1}, \qquad \text{ha} \qquad p \geqslant 0 \text{ páros}$$

$$D_{m,p+1}\Psi_{i_m}^{n} = \frac{c_{m,p+1}(nk)}{2} [\delta\Psi_{i_m}^{n}(|\delta\Psi_{i_m}^{n}|^p + (\delta\Psi_{i_m}^{n})^p) -$$

$$- \delta\Psi_{i_m-1}^{n}(|\delta\Psi_{i_m-1}^{n}|^p - (\delta\Psi_{i_m-1}^{n})^p)]$$

ha $p \geq 1$ páratlan, ahol $\delta \Psi^n_{i_m} = (\Psi^n_{i_1,\ldots,i_m+1,\ldots,i_s} - \Psi^n_{i_1,\ldots,i_m,\ldots,i_s})/h_m$.

Tekintsük a következő véges differencia sémát:

$$(3.2) \qquad (u^{n+1}_{i_m} - u^n_{i_m})/k = \sum_{m=1}^{s} D_{m,p+1} u^n_{i_m}, \qquad u^0_{i_1,\ldots,i_s} = u^0(i_1 h_1, \ldots, i_s h_s)$$

(ahol $u^n_{i_m} \equiv u^n_{i_1,\ldots,i_s}$) amely analóg az (1.7)-(1.2) problémával.

3.1 Tétel. *Legyenek* $\Theta_m = \dfrac{k}{h_m}$; C_0; $C_{1,m}$; $C_{2,m,p}$; $C_{3,m}$ *a 2.1 tételben definiált konstansok.*

Legyen u *a (3.2) véges differencia séma megoldása.*

Tegyük fel, hogy

$$(p+1)\Theta_m \cdot C_{m,p+1}(C_{1,m})^p \leq \frac{1}{s_{p+1}}, \qquad m = 1, \ldots, s; \qquad p = 0, \ldots, r-1;$$

és s_{p+1} *a nem azonosan* 0 $c_{m,p+1}(t)$*-k száma, valamint* $C_{m,p+1} = \sup_t |c_{m,p+1}(t)|$.

Ekkor u *kielégíti a (2.4), (2.5), (2.6), (2.7) becsléseket, és*

$$h_1 h_2 \ldots h_s \sum_{i_1,\ldots,i_s} (u^{n+1}_{i_m} - u^n_{i_m})/k \leq C_{6,p};$$

ahol

$$C_{6,p} = 2 s_{p+1} C C_1 C_{2,p};$$

és

$$C = \max_{m,p} |C_{m,p+1}|; \qquad C_1 = \max_m |C_{1,m}|; \qquad C_{2,p} = \max_m |C_{2,m,p}|.$$

Bizonyítás. Legyen $p \geq 1$ páratlan, $v^n_{i_m} = \delta u^n_{i_m}$ és $v^{n+1}_{i_m} = \delta u^{n+1}_{i_m}$. Ekkor (3.2)-ből következik:

$$(v^{n+1}_{i_m} - v^n_{i_m})/k - \sum_{m=1}^{s} (D_{m,p+1} u^n_{i_m+1} - D_{m,p+1} u^n_{i_m})/h_m = 0,$$

azaz

$$(3.3) \qquad v^{n+1}_{i_m} = \sum_{m=1}^{s} \left(\frac{1}{s_p} - 2\Theta_m \frac{c_{m,p+1}(nk)}{2} |v^n_{i_m}|^p \right) v^n_{i_m} + \Theta_m \frac{c_{m,p+1}(nk)}{2} *$$

$$* (|v^n_{i_m+1}|^p + (v^n_{i_m+1})^p) v^n_{i_m+1} + \Theta_m \frac{c_{m,p+1}(nk)}{2} (|v^n_{i_m-1}|^p - (v_{i_m-1})^p) v^n_{i_m-1}.$$

Legyen $\lambda_{p,m} = \dfrac{\Theta_m \, c_{m,p+1}(nk)}{2}$,

$$f_{1,m,p}(y) = \left(\frac{1}{s_p} - 2\lambda_{p,m} |y|^p\right) y ,$$

$$f_{2,m,p}(y) = \lambda_{p,m} (|y|^p + y^p) y ,$$

$$f_{3,m,p}(y) = \lambda_{p,m} (|y|^p - y^p) y .$$

Igy (3.3)-at írjuk ilyen formában:

$$v_{i_m}^{n+1} = \sum_{m=1}^{s} f_{1,m,p}(v_{i_m}^n) + f_{2,m,p}(v_{i_m+1}^n) + f_{3,m,p}(v_{i_m-1}^n) .$$

$f_{1,m,p}$; $f_{2,m,p}$; $f_{3,m,p}$ monoton nemcsökkenő függvénye y-nak $2(p+1)\lambda_{p,m} y^p \leqslant \dfrac{1}{s_p}$ esetén.

Ugyanis mivel $y_{i+1} > y_i$, és ha $y_{i+1} \geqslant 0$, $y_i \geqslant 0$ vagy $y_{i+1} \leqslant 0$, $y_i \leqslant 0$, akkor $f_{1,m,p}$ deriválásából adódik a feltétel. Ha $y_{i+1} \geqslant 0$, $y_i \leqslant 0$, akkor

$$f_{1,m,p}(y_{i+1}) - f_{1,m,p}(y_i) = y_{i+1}\left(\frac{1}{s_p} - 2\lambda_{m,p} y_{i+1}^p\right) - y_i\left(\frac{1}{s_p} - 2\lambda_{m,p} |y_i|^p\right) \geqslant 0$$

nyilvánvalóan. $f_{2,m,p}$ és $f_{3,m,p}$ a 0-ban is differenciálható, így ezekre is könnyen adódik a feltétel. Legyen $V = \sup_{i_m} v_{i_m}^n$, ekkor $f(-V) \leqslant v_{i_m}^{n+1} \leqslant f(V)$ minden i_m-re, ahol

$$f(y) \equiv \sum_{m=1}^{s} f_{1,m,p} + f_{2,m,p} + f_{3,m,p} \equiv y$$

Innen $|v_{i_m}^{n+1}| \leqslant V$, és (2.5) következik. $i_1, \dots, i_s$-re való szummázással:

$\sum_{i_1,\dots,i_s} |v_{i_m}^{n+1}|^p \leqslant \sum_{i_1,\dots,i_s} |v_{i_m}^n|^p$, és (2.6) igazolt.

(3.2)-ből következik, hogy

$$u_{i_m}^{n+1} = \sum_{m=1}^{s} \left[\frac{1}{s_p} - \lambda_{p,m}(|\delta u_{i_m}^n|^p + (\delta u_{i_m}^n)^p) - \lambda_{p,m}(|\delta u_{i_m-1}^n|^p - (\delta u_{i_m-1}^n)^p)\right] u_{i_m}^n +$$

$$+ \lambda_{p,m}(|\delta u_{i_m}^n|^p + (\delta u_{i_m}^n)^p) u_{i_m+1}^n + \lambda_{p,m}(|\delta u_{i_m-1}^n|^p - (\delta u_{i_m-1}^n)^p) u_{i_m-1}^n$$

$u_{i_m}^n$, $u_{i_m+1}^n$, $u_{i_m-1}^n$ együtthatói nemnegatívak, és összegük 1, így (2.4)-et kapjuk.

(3.3)-at írjuk ilyen alakban:

$$(3.4)\qquad v_{i_m}^{n+1} = v_{i_m}^n + \sum_{m=1}^{s} (f_{2,m,p}(v_{i_m+1}^n) - f_{2,m,p}(v_{i_m}^n)) - (f_{3,m,p}(v_{i_m}^n) - f_{3,m,p}(v_{i_m-1}^n)) =$$

$$= v_{i_m}^n + \sum_{m=1}^{s} \alpha_{p,i_m+1}(v_{i_m+1}^n - v_{i_m}^n) - \beta_{p,i_m}(v_{i_m}^n - v_{i_m-1}^n),$$

ahol

$$\alpha_{m,i_m+1} = \frac{f_{2,m,p}(v_{i_m+1}^n) - f_{2,m,p}(v_{i_m}^n)}{v_{i_m+1}^n - v_{i_m}^n} \quad \text{és} \quad \beta_{p,i_m} = \frac{f_{3,m,p}(v_{i_m}^n) - f_{3,mp}(v_{i_m-1}^n)}{v_{i_m}^n - v_{i_m-1}^n}.$$

Legyen $w_{i_m}^n = (v_{i_m}^n - v_{i_m-1}^n)/h_m$. (3.4)-ből következik, hogy:

$$w_{i_m}^{n+1} = w_{i_m}^n + \sum_{m=1}^{s} (\alpha_{p,i_m+1} w_{i_m+1}^n - \alpha_{p,i_m} w_{i_m}^n) - (\beta_{p,i_m} w_{i_m}^n - \beta_{p,i_m-1} w_{i_m-1}^n) =$$

$$= \sum_{m=1}^{s} \left(\frac{1}{s_p} - \alpha_{p,i_m} - \beta_{p,i_m} \right) w_{i_m}^n + \alpha_{p,i_m+1} w_{i_m+1}^n + \beta_{p,i_m-1} w_{i_m-1}^n.$$

Mivel $w_{i_m}^n$, $w_{i_m+1}^n$, $w_{i_m-1}^n$ együtthatói nemnegatívak:

$$|w_{i_m}^{n+1}| \leqslant \sum_{m=1}^{s} \left(\frac{1}{s_p} - \alpha_{p,i_m} - \beta_{p,i_m} \right) |w_{i_m}^n| + \alpha_{p,i_m+1} |w_{i_m+1}^n| + \beta_{p,i_m-1} |w_{i_m-1}^n|.$$

Majd összegezve:

$$\sum_{i_1,\ldots,i_s} |w_{i_m}^{n+1}| \leqslant \sum_{i_1,\ldots,i_s} |w_{i_m}^n|,$$

amiből (2.7) következik. Végül

$$h_1 h_2 \ldots h_s \sum_{i_1,\ldots,i_s} (u_{i_m}^{n+1} - u_{i_m}^n)/k = h_1 h_2 \ldots h_s \sum_{i_1,\ldots,i_s} \sum_{m=1}^{s_p} |D_{m,p+1} u_{i_m}^n| \leqslant$$

$$\leqslant h_1 h_2 \ldots h_s \sum_{i_1,\ldots,i_s} \sum_{m=1}^{s_p} c_{m,p+1}(nk) C_{1,m} (|\delta u_{i_m}^n|^p + (\delta u_{i_m-1}^n)^p) \leqslant 2 s_p C C_1 C_2 .$$

Páros p-re is hasonlóan megy a bizonyítás, hiszen ekkor $f_{3,m,p} \equiv 0$.

4. NUMERIKUS SÉMA AZ ÁLTALÁNOS (1.1) EGYENLETRE STABILITÁSI BECSLÉSEK

Legyen B_{h_m}; $D_{m,p}$ az előzőekben definiált operátor, μ_p pozitív szám, és $k_p = \dfrac{k}{\mu_p}$. Ekkor adott $u^n_{i_1,\dots,i_s}$-ből minden $i_1,\dots,i_s$-re kiszámolható $u^{n+1}_{i_1,\dots,i_s}$ a közbeeső $u^{n,q_j}_{i_1,\dots,i_s}$ függvények segítségével:

$$u^{n,0}_{i_1,\dots,i_s} \equiv u^n_{i_1,\dots,i_s},$$

$$(u^{n,q_1+1}_{i_1,\dots,i_s} - u^{n,q_1}_{i_1,\dots,i_s})/k_1 - \sum_{m=1}^{\bar{s}} B_{h_m} u^{n,q_1}_{i_1,\dots,i_s} = 0, \quad 0 \leq q_1 \leq \mu_1 - 1,$$

$$(u^{n,q_2+1}_{i_1,\dots,i_s} - u^{n,q_2}_{i_1,\dots,i_s})/k_2 - \sum_{m=1}^{s_1} D_{m,1} u^{n,q_2}_{i_1,\dots,i_s} = 0, \quad \mu_1 \leq q_2 \leq \mu_1 + \mu_2 - 1,$$

$$(4.1) \qquad \vdots$$

$$(u^{n,q_r+1}_{i_1,\dots,i_s} - u^{n,q_r}_{i_1,\dots,i_s})/k_r - \sum_{m=1}^{s_r-1} D_{m,r-1} u^{n,q_r}_{i_1,\dots,i_s} = 0, \quad \sum_{j=1}^{r-1} u_j \leq q_r \leq \sum_{j=1}^{r} u_j - 1,$$

$$(u^{n+1}_{i_1,\dots,i_s} - u^{n,q_r+1}_{i_1,\dots,i_s})/k - \sum_{m=1}^{s_r} D_{m,r} u^{n,q_r+1}_{i_1,\dots,i_s} = 0, \quad q_{r+1} = \sum_{j=1}^{r} u_j.$$

A sémát $u^0_{i_1,\dots,i_s} = u^0(i_1h_1,\dots,i_sh_s)$-el kezdjük.

4.1 Tétel. *Legyen* u *a (4.1) differencia rendszer megoldása. Tegyük fel:*

(4.2) $\quad 2M_m(k_1/h_m^2)\bar{s} \leq 1 \qquad m = 1,\dots,\bar{s};$

(4.3) $\quad \left(\dfrac{k_p}{h_m}\right)(p+1)\,C_{m,p+1}(C_{1,m})^p s_{p+1} \leq 1 \qquad m = 1,\dots,s_{p+1}; \quad p = 0,\dots,r-2;$

(4.4) $\quad \left(\dfrac{k}{h_m}\right) rC_{m,r}(C_{1,m})^{r-1} s_r \leq 1 \qquad m = 1,\dots,s_r.$

Ekkor u *kielégiti a (2.4), (2.5), (2.6), (2.7) becsléseket, és*

$$h_1h_2\dots h_s \sum_{i_1,\dots,i_s} |(u^{n+1}_{i_1,\dots,i_s} - u^n_{i_1,\dots,i_s})/k| \leq C_7 \qquad \textit{minden } n\textit{-re},$$

ahol $C_7 = rC_6 + M\bar{s}\,C_3$, *és* $C_6 = \max_p C_{6,p}$.

Bizonyítás. A becslések (2.4)-től (2.7)-ig rögtön következnek a 2.1 és 3.1 tételből. Másrészt (jelölés: $u^n_{i_1,\ldots,i_s} \equiv u^n_i$):

$$\left|\frac{u_i^{n+1}-u_i^n}{k}\right| \leqslant \left|\frac{u_i^{n+1}-u_i^{n,q_r+1}}{k}\right| + \frac{k_r}{k}\sum_{q_{p_m}}\left|\frac{u_i^{n,q_r+1}-u_i^{n,q_r}}{k_{p_m}}\right| + \ldots + \frac{k_1}{k}\sum_{q_1}\left|\frac{u_i^{n,q_1+1}-u_i^{n,q_1}}{k_1}\right| =$$

$$= \sum_{m=1}^{s_r} |D_{m,r} u_i^{n,q_r+1}| + \frac{1}{\mu_r}\sum_{q_r}\sum_{m=1}^{s_{r-1}} |D_{m,r-1} u_i^{n,q_r}| + \ldots + \frac{1}{\mu_1}\sum_{q_1}\sum_{m=1}^{\bar{s}} |B_{h_m} u_i^{n,q_1}| .$$

Innen

$$h_1,\ldots,h_s \sum_{i_1,\ldots,i_s} |(u^{n+1}_{i_1,\ldots,i_s} - u^n_{i_1,\ldots,i_s})/k| \leqslant C_{6,r} + \frac{1}{\mu_r}\sum_{q_r} C_{6,r-1} + \ldots$$

$$+ \frac{1}{\mu_1}\sum_{q_1} M\bar{s}\, C_3 \leqslant rC_6 + M\bar{s}\, C_3 = C_7 .$$

5. KOMPAKTSÁGI TÉTELEK

Differencia sémánk konvergenciájának bizonyításához szükségünk van két kompaktsági tételre.

5.1 Tétel. *Legyen V a következő függvények tere:*

$$V = \left\{z(x,t) : z \in \mathscr{L}^\infty(\mathscr{H});\ \frac{\partial z}{\partial x_i} \in \mathscr{L}^\infty(\mathscr{H}),\ i = 1,\ldots,s;\ \frac{\partial z}{\partial t} \in \mathscr{L}^\infty(0,\infty;\ \mathscr{L}^1(\mathscr{R}^s))\right\} ,$$

ahol általánositott deriváltakról van szó, ld. pl. [2]-ben.

A norma legyen:

$$\|z(x,t)\|_V = \|z\|_{\mathscr{L}^\infty(\mathscr{H})} + \sum_{i=1}^{s}\left\|\frac{\partial z}{\partial x_i}\right\|_{\mathscr{L}^\infty(\mathscr{H})} + \left\|\frac{\partial z}{\partial t}\right\|_{\mathscr{L}^\infty(0,\infty;\ \mathscr{L}^1(\mathscr{R}^s))} .$$

Ekkor $V \subset C^0(\bar{\mathscr{H}})$, és V bármely végtelen korlátos részhalmazából kiválasztható egyenletesen konvergens részsorozat $\bar{\mathscr{H}}$ bármely korlátos résztartományán.

Bizonyítás. Legyen $\rho(x_1,\ldots,x_s) \in C^\infty(\mathscr{R}^s)$, melyre $\rho(x) \geqslant 0$; $\rho(x) \equiv 0$, ha $|x_i| \geqslant 1$ $(i = 1,\ldots,s)$; $\int_{\mathscr{R}^s} \rho(x)dx_1,\ldots,dx_s = 1$.

Legyen $\rho_n(x) = n\rho(nx_1,\ldots,nx_s)$, és minden $z \in V$-re

$$(z * \rho_n)(x,t) = \int_{\mathscr{R}^s} z(x_1-\sigma_1,\ldots,x_s-\sigma_s,t)\rho_n(\sigma_1,\ldots,\sigma_s)d\sigma_1,\ldots,d\sigma_s .$$

Ekkor $z * \rho_n \in C^0(\bar{\mathscr{H}})$ (felhasználva a Szoboljev-lemmát: [2], §.8.), ugyanis

(5.1) $\|\partial(z * \rho_n)/\partial x_i\|_{\mathscr{L}^\infty(\mathscr{H})} \leq \|\partial z/\partial x_i\|_{\mathscr{L}^\infty(\mathscr{H})}$ $\quad i = 1, \dots, s;$

és

(5.2) $\left\| \frac{\partial}{\partial t}(z * \rho_n) \right\|_{\mathscr{L}^\infty(\mathscr{H})} = \left\| \frac{\partial z}{\partial t} * \rho_n \right\|_{\mathscr{L}^\infty(\mathscr{H})} \leq \|\rho_n\|_{\mathscr{L}^\infty} \left\| \frac{\partial z}{\partial t} \right\|_{\mathscr{L}^\infty(0,\infty;\, \mathscr{L}^1(\mathscr{R}^s))}$

De $z * \rho_n$ $\mathscr{H}$-ban egyenletesen konvergál z-hez, mert z egyenletesen folytonos x_i-re nézve, így z folytonos. Legyen most $\mathscr{F}$ a V-ben egyenletesen korlátos $z \in V$ függvényeknek egy végtelen családja. Ekkor z és $\frac{\partial z}{\partial x_i}$ $(i = 1, \dots, s)$ egyenletesen korlátos $\mathscr{L}^\infty(\mathscr{H})$-ban, és így kiválaszthatunk olyan z_r sorozatot, hogy $z_r \longrightarrow Z$ és $\frac{\partial z_r}{\partial x_i} \longrightarrow \frac{\partial Z}{\partial x_i}$ az $\mathscr{L}^\infty(\mathscr{H})$ gyenge* topológiájában, ahol Z egy bizonyos függvény.

Legyen G a $\mathscr{H}$ tetszőleges résztartománya.

(5.3) $$\|z_r - Z\|_{\mathscr{L}^\infty(G)} \leq \|z_r - z_r * \rho_n\|_{\mathscr{L}^\infty(G)} + \|z_r * \rho_n - Z * \rho_n\|_{\mathscr{L}^\infty(G)} + \\ + \|Z * \rho_n - Z\|_{\mathscr{L}^\infty(G)}\,.$$

Legyen $\epsilon > 0$ adott. A z_r és Z függvények ekvifolytonosak x_i-re nézve, így választhatunk olyan n-et, hogy

(5.4) $\|z_r - z_r * \rho_n\|_{\mathscr{L}^\infty(\mathscr{H})} < \epsilon/3$ $\quad$ minden r-re,

$\|Z - Z * \rho_n\|_{\mathscr{L}^\infty(\mathscr{H})} < \epsilon/3\,.$

Legyen n fix, ekkor (5.1) és (5.2) miatt a $z_r * \rho_n$ függvények ekvifolytonosak $\mathscr{H}$-ban (x_i-re és t-re nézve). Így kiválaszthatunk egy részsorozatot, még mindig $z * \rho_n$-val jelölve, amely egyenletesen konvergál G-ben, amint $r \longrightarrow \infty$. A határérték szükségképpen $Z * \rho_n$, ezért

(5.5) $\|z_r * \rho_n - Z * \rho_n\|_{\mathscr{L}^\infty(\mathscr{H})} < \epsilon/3$ $\quad$ elég nagy r-re.

A tétel most már azonnal következik (5.3), (5.4) és (5.5)-ből. A következő tétel Aubin [3] egyik tételének speciális esete.

5.2 Tétel. *Legyen $\Omega^s \subset \mathscr{R}^s$ s dimenziós korlátos tartomány, $T > 0$ véges szám, $A \equiv \Omega^s x(0,T)$. Legyen W a következő függvények tere:*

$$W = \left\{ z(x,t) : z \in \mathscr{L}^\infty(A),\ \operatorname*{Var}_{x_k} z \in \mathscr{L}^\infty(0,T;\, \mathscr{L}^1(\overline{\mathscr{R}}_k));\ \frac{\partial z}{\partial t} \in \mathscr{L}^\infty(0,T; H^{-2}(\Omega^s));\ k = 1, \dots, s \right\},$$

ahol $H^{-2}(\Omega^s)$ *az a Szoboljev-tér, amely a* $H_0^2(\Omega^s)$ *duálisa. Definiáljuk a következő normát:*

$$\|z(x,t)\|_W = \|z\|_{\mathscr{L}^\infty(A)} + \sum_{k=1}^{s} \|\operatorname*{Var}_{x_k} z\|_{\mathscr{L}^\infty(0,T;\, \mathscr{L}^1(\bar{\mathscr{R}}_k))} + \left\| \frac{\partial z}{\partial t} \right\|_{\mathscr{L}^\infty(0,T;H^{-2}(\Omega^s))} .$$

Ekkor W *bármely végtelen korlátos részhalmazából kiválasztható* $\mathscr{L}^p(A)$*-ban erősen konvergens részsorozat, ahol* $1 \leqslant p < \infty$ *tetszőleges.*

6. KONVERGENCIA

Tegyük fel, hogy $k_1, \ldots, k_p, k$ a $h_1, \ldots, h_s$-nek függvénye. Jelöljük $u_h \equiv u^n_{i_1,\ldots,i_s}$-el a (4.1) véges differenciarendszer megoldását, és terjesszük ki azt az egész $\mathscr{H}$ féltérre a következőképpen. Tekintsük az s dimenziós $I((i_1 + \Theta_1)h_1, \ldots, (i_s + \Theta_s)h_s)$ (ahol $\Theta_1, \ldots, \Theta_s \in [0,1]$) kockát. Válasszuk ki a kocka $C(i_1 h_1, \ldots, i_s h_s)$ csúcsát. Vegyük a kocka összes lapját, melynek nem csúcsa C. Ekkor C és ezek a lapok sorra s-dimenziós zárt szimplexeket alkotnak, melyeknek nincs közös belső pontjuk, és egyesítésük az egész kocka. Ismeretes az is, hogy egy szimplex bármely belső (vagy határ-) pontja súlyponti koordináták segítségével egyértelműen kifejezhető a szimplex csúcsaival. Ilymódon ha I egy pontja az S_j szimplexbe (vagy határára) esik, akkor legyen

$$u_h((i_1 + \Theta_1)h_1, \ldots, (i_s + \Theta_s)h_s, (n + \Theta')k) = \sum_{j=1}^{s+1} \alpha_j u_j(C_j, nk) ,$$

ahol $\sum_{j=1}^{s+1} \alpha_j = 1$, $0 \leqslant \alpha_j \leqslant 1$, α_j lineáris a Θ_l-ekben, valamint C_j-k az S_j szimplex csúcsai; $\Theta' \in [0,1]$; azaz u_h darabonként lineáris x-ben, konstans t-ben.

6.1 Tétel. *Legyenek* $\{h_1\}, \ldots, \{h_s\}$ *0-hoz tartó sorozatok. Tegyük fel, hogy a (4.2), (4.3), (4.4) feltételek kielégülnek minden* h_m*-re* ($m = 1, \ldots, s$). *Ekkor létezik* $\{h_1\}$*-nek,* ..., $\{h_s\}$*-nek olyan részsorozata, hogy* u_h *egyenletesen konvergál bármely* $G \subset \mathscr{H}$ *korlátos résztartományon az (1.1)-(1.2) probléma egy* U *megoldásához, és* $\dfrac{\partial u_h}{\partial x_m}$ *konvergál* $\dfrac{\partial U}{\partial x_m}$*-hez* $\mathscr{L}^p(G)$*-ben erősen* ($m = 1, \ldots, s$) *minden* $1 \leqslant p < \infty$*-re.*

A tételt lépésenként bizonyitjuk. Először néhány lemmát igazolunk.

6.1 Lemma. *Definiáljuk* $\tilde{u}_h$*-t a következőképpen:*

$$\tilde{u}_h(x,(n + \Theta/2)k) = u_h(x,(n + \Theta/2)k) = u_h(x,nk) ,$$

$$\tilde{u}_h\left(x,\left(n + \frac{1}{2} + \frac{\Theta}{2}\right) k\right) = (1 - \Theta)u_h(x,nk) + \Theta u_h(x,(n + 1)k) .$$

$0 \leqslant \Theta < 1$*-re. Ekkor létezik* $\{h_1\}$*-nek,* ..., $\{h_s\}$*-nek olyan részsorozata, és olyan* U *függvény, hogy* $U \in C^0(\bar{\mathscr{H}}) \cap \mathscr{L}^\infty(\mathscr{H})$, $U(x,0) = u^0(x)$ *minden* $x \in \mathscr{R}^s$*-re,* $\dfrac{\partial U}{\partial x_m} \in \mathscr{L}^\infty(\mathscr{H})$, $\tilde{u}_h \longrightarrow U$

G-ben egyenletesen, és $\dfrac{\partial \tilde{u}}{\partial x_m} \longrightarrow \dfrac{\partial U}{\partial x_m}$ *$\mathscr{L}^p(G)$-ben minden korlátos $G \subset \mathscr{H}$-re.*

Bizonyítás. A 4.1 tétel becsléseit írhatjuk ilyen formában:

$$\|\tilde{u}_h\|_{\mathscr{L}^\infty(\mathscr{H})} \leqslant C_0\,, \tag{6.1}$$

$$\|\partial\tilde{u}_h/\partial x_m\|_{\mathscr{L}^\infty(\mathscr{H})} \leqslant C_{1,m}\,, \tag{6.2}$$

$$\|\partial\tilde{u}_h/\partial x_m\|_{\mathscr{L}^\infty(0,\infty;\,\mathscr{L}^p(\mathscr{R}^s))} \leqslant C_{2,m,p}\,, \tag{6.3}$$

$$\left\|\operatorname*{Var}_{x_m}\left(\frac{\partial u_h}{\partial x_m}\right)\right\|_{\mathscr{L}^\infty(0,\infty;\,\mathscr{L}^1(\bar{\mathscr{R}}_m))} \leqslant C_{3,m}\,, \tag{6.4}$$

$$\|\partial\tilde{u}_h/\partial t\|_{\mathscr{L}^\infty(0,\infty;\,\mathscr{L}^1(\mathscr{R}^s))} \leqslant 2C_7\,. \tag{6.5}$$

(6.5)-ből következik, hogy

$$\left\|\frac{\partial}{\partial t}\left(\frac{\partial}{\partial x_m}\,\tilde{u}_h\right)\right\|_{\mathscr{L}^\infty(0,\infty;H^{-2}(\Omega^s))} \leqslant C_7'(\Omega^s)\,, \tag{6.6}$$

ahol $\Omega^s \subset \mathscr{R}^s$ tetszőleges véges tartomány, és $C_7'(\Omega^s)$ egy Ω^s-től függő konstans (mert $\left\|\dfrac{\partial z}{\partial x_m}\right\|_{H^{-2}(\Omega)} \leqslant C'(\Omega)\|z\|_{\mathscr{L}^1(\mathscr{R}^s)}$ minden $z \in \mathscr{L}^1(\mathscr{R}^s)$ esetén). (6.1), (6.2) és (6.5) alapján az 5.1 tételből következik, hogy létezik a $\{h_1\}$-nek, . . . , $\{h_s\}$-nek olyan részsorozata és egy olyan $U \in \mathscr{L}^\infty(\mathscr{H}) \cap C^0(\overline{\mathscr{H}})$, hogy $\tilde{u}_h$ konvergál U-hoz egyenletesen a G-ben.
(6.2), (6.4) és (6.6)-ból az 5.2 tétel miatt következik, hogy létezik az előbbi részsorozatnak egy-egy olyan részsorozata, hogy $\dfrac{\partial \tilde{u}_h}{\partial x_m}$ konvergál $\dfrac{\partial U}{\partial x_m} \in \mathscr{L}^\infty(\mathscr{H})$-hez erősen az $\mathscr{L}^p(G)$-ben.
(A második határérték szükségképpen $\dfrac{\partial U}{\partial x_m}$, mivel $\tilde{u}_h$-nak U-hoz való konvergenciájából következik $\dfrac{\partial \tilde{u}_h}{\partial x_m}$-nek $\dfrac{\partial U}{\partial x_m}$-hez való konvergenciája a disztribúciók terében.) Az a tény, hogy U kielégíti az (1.2) kezdeti feltételt, következik az egyenletes konvergenciából.
Az alábbiakban mindig a kiválasztott részsorozatokhoz tartozó h_m értékeket fogjuk tekinteni.

6.2 Lemma. *$u_h \longrightarrow U$ egyenletesen G-ben, és* $\dfrac{\partial u}{\partial x_m} \longrightarrow \dfrac{\partial U}{\partial x_m}$ *$(m = 1, \dots, s)\,\mathscr{L}^p(G)$-ben.*

Bizonyítás. Legyen $\tau_{k/2}$ az eltolási operátor: $\tau_{k/2}\Psi(x,t) = \Psi\left(x, t + \dfrac{k}{2}\right)$,

$$\|u_h - U\|_B \leqslant \|\tilde{u}_h - U\|_B + \|\tilde{u}_h - \tau_{k/2}U\|_B \leqslant 2\|\tilde{u}_h - U\|_B + \|U - \tau_{k/2}U\|_B\,,$$

ahol

$$\|\Psi\|_B = \max_G |\Psi(x,t)| + \sum_{m=1}^{s} \left\| \frac{\partial \Psi}{\partial x_m} \right\|_{\mathcal{L}^p(G)} .$$

De $\|\widetilde{u}_h - U\|_B \longrightarrow 0$ a 6.1 lemma szerint, és $\|U - \tau_{k/2} U\|_B \longrightarrow 0$, mert U és $\dfrac{\partial U}{\partial x_m}$ is folytonos $\mathcal{L}^p(G)$-ben (ld. Szoboljev [2]).

6.3 Lemma. *Terjesszük ki az* $u^{n,q_k}_{i_1,\ldots,i_s}$ *függvényeket az egész* $\mathcal{H}$ *féltérre a 6. pont elején látott módon:*

$$u_h^{(q_i)}((i_1 + \Theta_1)h_1, \ldots, (i_s + \Theta_s)h_s, (n + \Theta')k) = \sum_{j=1}^{s+1} \alpha_j u_j^{n,q_i} ,$$

ahol $\Theta_1, \ldots, \Theta_s, \Theta' \in [0,1]$, $\sum_{j=1}^{s+1} \alpha_j = 1$, $0 \leqslant \alpha_j \leqslant 1$.

Ekkor $\max\limits_{0 \leqslant q_k \leqslant q_{r+1}} \|u_h^{(q_k)} - U\|_B \longrightarrow 0$, *ha* $h_m \longrightarrow 0$; $m = 1, \ldots, s$.

Bizonyítás. Legyen az $\widetilde{u}_h^{(q_k)}$ függvény a következő:

$$\widetilde{u}_h^{(q_k)}(x,t) = u_h^{(q_k)}(x,t) = u_h^{(q_k)}(x,nk), \text{ ha } \left(n + \frac{1}{3}\right)k \leqslant t \leqslant \left(n + \frac{2}{3}\right)k ,$$

$$\widetilde{u}_h^{(q_k)}(x,(n + \Theta/3)k) = (1 - \Theta)u_h(x,nk) + \Theta u_h^{(q_k)}(x,nk) ,$$

$$\widetilde{u}_h^{(q_k)}\left(x, \left(n + \frac{2}{3} + \frac{\Theta}{3}\right)k\right) = (1 - \Theta)u_h^{(q_k)}(x,nk) + \Theta u_h(x,(n + 1)k) ,$$

ahol $0 \leqslant \Theta < 1$.

Az $\widetilde{u}_h^{(q_i)}$ függvények kielégítik ugyanazokat a becsléseket mint $\widetilde{u}_h$, azaz a (6.1)-(6.5) becsléseket ($2C_7$ helyett $3C_7$-el).

Legyen d olyan egész, hogy $1 \leqslant d \leqslant q_{r+1}$, és

$$\|\widetilde{u}_h^{(d)} - U\|_B = \sup_{1 \leqslant q_i \leqslant q_{r+1}} \|\widetilde{u}_h^{(q_i)} - U\|_B .$$

Az 5.1 és 5.2 tételből következik, hogy létezik olyan U^* függvény, amely ugyanolyan tulajdonságú, mint U, és amelyre $\|\widetilde{u}_h^{(d)} - U^*\|_B \longrightarrow 0$ a $\{h_1\}, \ldots, \{h_s\}$ bizonyos részsorozatára. De $u_h(x,nk) = \widetilde{u}_h^{(d)}(x,nk)$ minden $x \in \mathcal{R}^s$ és minden n esetén, amiből a 6.2 lemma szerint $U = U^*$. Hasonlóan, mint a 6.2 lemma bizonyításánál, kapjuk

$$\|u_h^{(d)} - U\| \leqslant 3\|\widetilde{u}_h^{(d)} - U\| + \|U - \tau_{k/3} U\| + \|U - \tau_{-k/3} U\| \longrightarrow 0 .$$

A 6.1 tétel bizonyítása. Most már csak azt kell igazolni, hogy a 6.1, 6.2 és 6.3 lemmákban előállított U függvény kielégíti az (1.3) integrál összefüggést. Összeadva a (4.1) egyenleteket, kapjuk ($u_i^{n,q_i} \equiv u^{n,q_i}_{i_1,\ldots,i_s}$, $\Phi_i^n \equiv \Phi^n_{i_1,\ldots,i_s}$)

$$(6.7)\quad \frac{u_i^{n+1}-u_i^n}{k}-\frac{1}{\mu_1}\sum_{q_1=0}^{\mu_1-1}\sum_{m=1}^{s}B_{h_m}u_i^{n,q_1}-\frac{1}{\mu_2}\sum_{q_2}\sum_{m=1}^{s}D_{m,1}u_i^{n,q_2}-\ldots-\sum_{q_r}\sum_{m=1}^{s}D_{m,r}u_i^{n,q_r+1}.$$

Legyen $\Phi\in\mathscr{D}(\mathscr{H})$. Szorozzuk meg (6.7)-et $h_1,\ldots,h_s k\,\Phi^n_{i_1,\ldots,i_s}$-el és összegezzünk minden i_j-re és n-re, parciális összegzést is alkalmazva kapjuk:

$$-h_1h_2\ldots h_sk\sum_{i_1,\ldots,i_s,n}u^n_{i_1,\ldots,i_s}\frac{\Phi^n_{i_1,\ldots,i_s}-\Phi^{n-1}_{i_1,\ldots,i_s}}{k}+$$

$$+\frac{1}{\mu_1}\sum_{q_1=0}^{\mu_1-1}h_1h_2\ldots h_sk\sum_{i_1,\ldots,i_s,n}\sum_{m=1}^{s}f^{n,q_1}_{m,i_m}\,\delta u^{n,q_1}_{i_m}\,\delta\Phi^n_{i_m}+$$

$$+\frac{1}{\mu_1}\sum_{q=0}^{\mu_1-1}h_1h_2\ldots h_sk\sum_{i_1,\ldots,i_s,n}\sum_{m=1}^{s}\Phi^n_{i_m+1}\delta f^{n,q_1}_{m,i_m}\,\delta u^{n,q_1}_{i_m}+$$

$$+\sum_{p=0}^{r-2}\frac{1}{\mu_{p+2}}\sum_{q_{p+2}}\sum_{m=1}^{s}h_1h_2\ldots h_sk\sum_{i_1,\ldots,i_s,n}\Big\{\frac{c_{m,p+1}(nk)}{2}[(\Phi^n_{i_m}+\Phi^n_{i_m+1})\cdot(\delta u^{n,q_p+1}_{i_m})^{p+1}+$$

$$+h_m\,\delta u^{n,q_p+1}_{i_m}|\delta u^{n,q_p+1}_{i_m}|^p\,\delta\Phi^n_{i_m}]\Big\}+\sum_{m=1}^{s}h_1h_2\ldots h_sk\sum_{i_1,\ldots,i_s,n}\Big\{\frac{c_{m,q_r+1}(nk)}{2}[(\Phi^n_{i_m}+\Phi^n_{i_m+1})*$$

$$*(\delta u^{n,q_r+1}_{i_m})^r+h_m\,\delta u^{n,q_r+1}_{i_m}|\delta u^{n,q_r+1}_{i_m}|^{r-1}\delta\Phi^n_{i_m}]\Big\}.$$

Ekkor a tétel következik a következő lemmából:

6.4 Lemma. *Ha* $h_1\longrightarrow 0,\ldots,h_s\longrightarrow 0$, akkor

$$(6.8)\quad h_1h_2\ldots h_sk\sum_{i_1,\ldots,i_s,n}u^n_{i_1,\ldots,i_s}\frac{\Phi^n_i-\Phi^{n-1}_i}{k}\longrightarrow\int_{\mathscr{H}}U\frac{\partial\Phi}{\partial t},$$

$$(6.9)\quad h_1h_2\ldots h_sk\sum_{i_1,\ldots,i_s,n}(\Phi^n_{i_m}+\Phi^n_{i_m+1})(\delta u^{n,q_j}_{i_m})^{p+1}\longrightarrow 2\int_{\mathscr{H}}\Phi\left(\frac{\partial U}{\partial x_m}\right)^{p+1},$$

$$(6.10)\quad h_1h_2\ldots h_sk\sum_{i_1,\ldots,i_s,n}h_m\,\delta u^{n,q_j}_{i_m}|\delta u^{n,q_j}_{i_m}|^p\,\delta\Phi^n_{i_m}\longrightarrow 0,$$

$$(6.11)\quad h_1h_2\ldots h_sk\sum_{i_1,\ldots,i_s,n}f^{n,q_1}_{m,i_m}\,\delta u^{n,q_1}_{i_m}\,\delta\Phi^n_{i_m}\longrightarrow\int_{\mathscr{H}}f_m\frac{\partial U}{\partial x_m}\frac{\partial\Phi}{\partial x_m},$$

$$(6.12)\qquad h_1h_2\ldots h_s k \sum_{i_1,\ldots,i_s,n} \Phi^n_{i_m+1}\,\delta f^{n,q_1}_{m,i_m}\,\delta u^n_{i_m} \longrightarrow \int_{\mathscr{H}} \left[\frac{\partial f_m}{\partial x_m} u_{x_m} + \frac{\partial f_m}{\partial u}(u_{x_m})^2\right]\Phi +$$

$$+\sum_{\underline{k}=1}^{N} - \left\{\frac{(-1)^{\underline{k}+1}}{(\underline{k}+1)!}\left[\frac{\partial^{\underline{k}} f_m}{\partial u^{\underline{k}}_{x_m}}(u_{x_m})^{\underline{k}+1} + \left(\frac{\partial}{\partial x_m}\left(\frac{\partial^{\underline{k}} f_m}{\partial u^{\underline{k}}_{x_m}}\right)\right)(u_{x_m})^{\underline{k}+1}\Phi +\right.\right.$$

$$\left.\left.+\left(\frac{\partial}{\partial u}\left(\frac{\partial^{\underline{k}} f_m}{\partial u^{\underline{k}}_{x_m}}\right)\right)(u_{x_m})^{\underline{k}+2}\Phi\right]\right\}.$$

Bizonyítás. Ezeknek az állításoknak igazolása hasonlóan történik, így közülük csak a legnehezebbet, a (6.12)-t mutatjuk meg.

A 6.3 lemma szerint minden $u_h^{(q_i)}$ függvény konvergál U-hoz, q_i-ben egyenletesen, ugyanolyan módon, mint u_h, ezért (6.12)-t elég bizonyítani $u_h^{(q_i)}$ helyett u_h-ra. Legyen

$$X \equiv h_1h_2\ldots h_s k \sum_{i_1,\ldots,i_s,n} \Phi^n_{i_1+1,\ldots,i_s+1}\,\delta f^n_{m,i_m}\,\delta_m u^n_{i_1,\ldots,i_s} \equiv X_1 + X_2 + X_3\,,$$

ahol

$$X_1 \equiv h_1h_2\ldots h_s k \sum_{i_1,\ldots,i_s,n} \Phi^n_{i+1}\left(\frac{f_m(x_{i_1},\ldots,x_{i_m+1},\ldots,x_{i_s},t^n,u^n_{i_m+1},\delta u^n_{i_m+1})}{h_m} - \right.$$

$$\left. - \frac{f_m(x_{i_1},\ldots,x_{i_m},\ldots,x_{i_s},t^n,u^n_{i_m+1},\delta u^n_{i_m+1})}{h_m}\right)\frac{u^n_{i_m+1}-u^n_{i_m}}{h_m}\,,$$

$$X_2 \equiv h_1h_2\ldots h_s k \sum_{i_1,\ldots,i_s,n} \Phi^n_{i+1}\left(\frac{f_m(x_{i_1},\ldots,x_{i_m},\ldots,x_{i_s},t^n,u^n_{i_m+1},\delta u^n_{i_m+1})}{h_m} - \right.$$

$$\left. - \frac{f_m(x_{i_1},\ldots,x_{i_m},\ldots,x_{i_s},t^n,u^n_{i_m},\delta u^n_{i_m+1})}{h_m}\right)\frac{u^n_{i_m+1}-u^n_{i_m}}{h_m}\,,$$

$$X_3 \equiv h_1h_2\ldots h_s k \sum_{i_1,\ldots,i_s,n} \Phi^n_{i+1}\left(\frac{f_m(x_{i_1},\ldots,x_{i_m},\ldots,x_{i_s},t^n,u^n_{i_m},\delta u^n_{i_m+1})}{h_m} - \right.$$

$$\left. - \frac{f_m(x_{i_1},\ldots,x_{i_m},\ldots,x_{i_s},t^n,u^n_{i_m},\delta u^n_{i_m})}{h_m}\right)\frac{u^n_{i_m+1}-u^n_{i_m}}{h_m}\,,$$

ahol $x_{i_j} = i_j h_j$, $x_{i_j+1} = (i_j+1)h_j$, $t^n = nk$, $\Phi^n_i \equiv \Phi^n_{i_1,\ldots,i_s}$.

De $\Phi^n_{i_1,\dots,i_m+1,\dots,i_s} \equiv \Phi^n_{i_1,\dots,i_m,\dots,i_s} + 0(h_m)$,

$$\frac{f_m(x_{i_1},\dots,x_{i_m+1},\dots,x_{i_s},t^n,u^n_{i_m+1},\delta u^n_{i_m+1}) - f_m(x_{i_1},\dots,x_{i_m},\dots,x_{i_s},t^n,u^n_{i_m+1},\delta u^n_{i_m+1})}{h_m} =$$

$$= \frac{\partial f_m}{\partial x_m}(x_{i_1},\dots,x_{i_m},\dots,x_{i_s},t^n,u^n_{i_m},\delta u^n_{i_m}) + \epsilon(h_m),$$

$$\frac{f_m(x_{i_1},\dots,x_{i_m},\dots,x_{i_s},t^n,u^n_{i_m+1},\delta u^n_{i_m+1}) - f_m(x_{i_1},\dots,x_{i_m},\dots,x_{i_s},t^n,u^n_{i_m},\delta u^n_{i_m+1})}{h_m} =$$

$$= \frac{\partial f}{\partial u}(x_{i_1},\dots,x_{i_m},\dots,x_{i_s},t^n,u^n_{i_m},\delta u^n_{i_m})\frac{u^n_{i_m+1} - u^n_{i_m}}{h_m} + \underline{\epsilon}(h_m),$$

ahol $\epsilon(h_m)$ és $\underline{\epsilon}(h_m) \longrightarrow 0$ egyenletesen a Φ tartóján, amint $h_m \longrightarrow 0$ (mert $\frac{\partial}{\partial x_m} f_m$, $\frac{\partial}{\partial u} f_m$ folytonos, és $(u^n_{i_m+1} - u^n_{i_m})/h_m$ egyenletesen korlátos). Ilymódon

$$(6.14)\qquad X_1 = h_1 h_2 \dots h_s k \sum_{i_1,\dots,i_s,n} (F_{h_m})^n_{i_m} \frac{u^n_{i_m+1} - u^n_{i_m}}{h_m} + \overline{\epsilon}(h_m),$$

$$(6.15)\qquad X_2 = h_1 h_2 \dots h_s k \sum_{i_1,\dots,i_s,n} (G_{h_m})^n_{i_m} \left(\frac{u^n_{i_m+1} - u^n_{i_m}}{h_m}\right)^2 + \epsilon'(h_m),$$

ahol $\qquad F_{h_m}(x,t) = \Phi(x,t)\frac{\partial}{\partial x_m} f_m(x,t,u_h(x,t),\delta u_h(x,t))$,

$$G_{h_m}(x,t) = \Phi(x,t)\frac{\partial}{\partial u} f_m(x,t,u_h(x,t),\delta u_h(x,t)).$$

Legyen $Q^n_{i_m}$ a következő kocka: $i_m h_m \leqslant x_{i_m} \leqslant (i_m + 1)h_m$, $m = 1,\dots,s$; $nk \leqslant t \leqslant (n+1)k$. A középérték szerint

$$\int_{Q^n_i} F_{h_m} \frac{\partial u_h}{\partial x_m} = F_{h_m}(\xi^n_{i_m}) \int_{Q^n_{i_m}} \frac{\partial u_h}{\partial x_m} = h_1 h_2 \dots h_s k F_{h_m}(\xi^n_{i_m}) \frac{u^n_{i_m+1} - u^n_{i_m}}{h_m},$$

ahol $\xi^n_i \equiv \xi^n_{i_1,\dots,i_s} \in Q^n_{i_m}$. Igy

$$(6.16)\qquad \int_{\mathscr{H}} F_{h_m} \frac{\partial u_h}{\partial x_m} = h_1 h_2 \dots h_s k \sum_{i_1,\dots,i_s,n} F_{h_m}(\xi^n_{i_m}) \frac{u^n_{i_m+1} - u^n_{i_m}}{h_m}.$$

A 6.2 lemma szerint F_{h_m}, illetve G_{h_m} $\mathscr{L}^p(G)$-ben konvergál a következő folytonos függvényhez:

$$F_m(x,t) = \Phi(x,t) \frac{\partial}{\partial x_m} f_m\left(x,t,U(x,t),\frac{\partial U(x,t)}{\partial x_m}\right) \text{-hez, illetve}$$

$$G_m(x,t) = \Phi(x,t) \frac{\partial}{\partial U} f_m\left(x,t,U(x,t),\frac{\partial U(x,t)}{\partial x_m}\right) \text{-hez}.$$

Ha $h_m \longrightarrow 0$; $m = 1, \dots, s$, akkor $|F_{h_m}(\xi^n_{i_m}) - (F_{h_m})^n_{i_m}| \longrightarrow 0$ $\mathscr{L}^p(G)$-ben.
Összehasonlítva (6.14)-et és (6.16)-ot kapjuk:

$$X_1 = \int_{\mathscr{H}} F_{h_m} \frac{\partial u_h}{\partial x_m} + \bar{\epsilon}(h_m).$$

Hasonló módon:

$$X_2 = \int_{\mathscr{H}} G_{h_m} \left(\frac{\partial u_h}{\partial x_m}\right)^2 + \epsilon'(h_m).$$

Mármost $F_{h_m} \longrightarrow F_m$, és $\frac{\partial u_h}{\partial x_m} \longrightarrow \frac{\partial U}{\partial x_m}$ erősen $\mathscr{L}^p(S)$-ben, ahol S a Φ tartója. Igy $X_1 \longrightarrow \int_{\mathscr{H}} F_m \frac{\partial U}{\partial x_m}$. Másrészt $\frac{\partial u_h}{\partial x_m}$ $\mathscr{L}^p(S)$-beli erős konvergenciájából – minden p-re, ha $1 \leqslant p < \infty$ – következik $\left(\frac{\partial u_h}{\partial x_m}\right)^j$ erős konvergenciája $\left(\frac{\partial U}{\partial x_m}\right)^j$-hez $\mathscr{L}^p(S)$-ben, ahol $j \geqslant 2$. Igy $X_2 \longrightarrow \int_{\mathscr{H}} G_m \left(\frac{\partial U}{\partial x_m}\right)^2$.

A továbbiakban felhasználjuk a következő összefüggést:

$$(6.17) \qquad \delta((\delta u_i)^r) = r(\delta u_i)^{r-1}\delta^2 u_i + \sum_{\underline{k}=1}^{r} c_{\underline{k}}(h\delta^2 u_i)^{\underline{k}},$$

ahol a $|c_{\underline{k}}| - k$ korlátosak, mivel $|\delta u_i|$ korlátos. Ugyanis

$$\delta((\delta u_i)^r) = \frac{\left(\frac{u_{i+1} - u_i}{h}\right)^r - \left(\frac{u_i - u_{i-1}}{h}\right)^r}{h} = \delta^2 u_i(a^{r-1} + a^{r-2}b + \dots + ab^{r-2} + b^{r-1}),$$

ahol

$$a = \frac{u_{i+1} - u_i}{h}, \qquad b = \frac{u_i - u_{i-1}}{h} = a - h\delta^2 u_i.$$

Esetünkben

$$\frac{\partial}{\partial x_m} u_h((i_1+\Theta_1)h_1,\dots,(i_m+\Theta_m)h_m,\dots,(i_s+\Theta_s)h_s)=\frac{u^n_{i_m+1}-u^n_{i_m}}{h_m},$$

$$\frac{\partial}{\partial x_m} u_h((i_1+\Theta_1)h_1,\dots,(i_m+\Theta_m-1)h_m,\dots,(i_s+\Theta_s)h_s)=\frac{u^n_{i_m+1}-u^n_{i_m}}{h_m}.$$

Mivel $\frac{\partial u_h}{\partial x_m}\longrightarrow\frac{\partial U}{\partial x_m}$ $\mathscr{L}^p(G)$-ben, így $h_m\delta^2 u_{i_m}\longrightarrow 0$ $\mathscr{L}^p(G)$-ben.

$$\frac{f_m(x_{i_1},\dots,x_{i_s},t^n,u^n_{i_m}\,\delta u^n_{i_m+1})-f_m(x_{i_1},\dots,x_{i_s},t^n,u^n_{i_m},\delta u^n_{i_m})}{h_m}=\frac{\partial}{\partial u_{x_m}}f_m(\dots,\delta u^n_{i_m})\delta^2 u^n_{i_m}+$$
$$+\frac{\partial^2}{\partial u^2_{x_m}}f_m(\dots,\xi^n_{i_m})h_m\,\delta^2 u^n_{i_m}.$$

Vegyük figyelembe, hogy

$$\delta\left(\frac{\partial f_m}{\partial u_{x_m}}\cdot(\delta u^n_{i_m})^2\right)=\delta\frac{\partial f_m}{\partial u_{x_m}}\cdot(\delta u^n_{i_m})^2+\frac{\partial f_m}{\partial u_{x_m}}(2(\delta u^n_{i_m})\delta^2 u_{i_m}+\epsilon_1(h_m)),\ \text{ahol}$$

$$\epsilon_1(h_m)\longrightarrow 0\quad \mathscr{L}^p(G)\text{-ben.}$$

Ekkor parciális összegzés után így alakul X_3:

$$\frac{1}{2}h_1h_2\dots h_s k\sum_{i_1,\dots,i_s,n}\frac{\partial f_m}{\partial x_m}(\delta u^n_{i_m})^2\delta\Phi^n_{i_m}-\frac{1}{2}h_1h_2\dots h_s k\sum_{i_1,\dots,i_s,n}\left(\frac{\partial^2 f_m}{\partial u_{x_m}\partial x_m}(\delta u^n_{i_m})^2+\right.$$
$$\left.+\frac{\partial^2 f_m}{\partial u_{x_m}\partial u}(\delta u^n_{i_m})^3\right)\Phi^n_i-\frac{1}{2}h_1h_2\dots h_s k\sum_{i_1,\dots,i_s,n}\frac{\partial^2 f_m}{\partial u^2_{x_m}}(\delta u^n_{i_m})^2(\delta^2 u^n_{i_m})\Phi^n_i+\epsilon_2(h_m),$$

ahol $\epsilon_2(h_m)$-mel a fellépő maradéktag-összegeket jelöltük, melyre $\epsilon_2(h_m)\longrightarrow 0$ $\mathscr{L}^p(G)$-ben. A fenti kifejezés első két tagja esetén a konvergencia bizonyítása az előzőekhez hasonlóan történik. Az utolsó tag esetében a (6.17) összefüggést alkalmazva a már ismert eljárást hajtjuk végre; és mindezt addig ismételjük, amíg $\frac{\partial^n f_m}{\partial u^n_{x_m}}\equiv 0$ lesz. Ilymódon (6.12) bizonyítását befejeztük.

Az (1.1)-(1.2) feladat általánosításáról, a konvergencia rendjéről, hibabecslésről egy későbbi cikkben lesz szó.

Irodalom

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Summary

A convergent finite difference scheme is described for some degenerate nonlinear parabolic equation in several variables.

In the paper [1] J.L. Graveleau and P. Jamet give the numerical solution of the equation

(1) $$\frac{\partial u}{\partial t} = f(x,t,u)\frac{\partial^2 u}{\partial x^2} + a\left(\frac{\partial u}{\partial x}\right)^2$$

with the initial condition

(2) $$u(x,0) = u^0(x)$$

if $f(x,t,u) \geqslant 0$, $a \geqslant 0$ is a constant, x is one-dimensional.

In this article – developing the methods adopted in [1] we give an explicit finite difference approximation for the equation (1.1)-(1.2) which is more general than (1)-(2).

At first we get stability estimates for the difference scheme based on the splitting of the operator A. Then we prove the convergence of the method by means of two compactness theorems.

Р е з ю м е

Сходящийся метод конечных разностей для некоторого многомерного нелинейного вырожденного уравнения параболического типа

В работе [1] Гравело и Жаме рассматривают численное решение уравнения

$$\frac{\partial u}{\partial t} = f(x, t, u) \frac{\partial^2 u}{\partial x^2} + a\left(\frac{\partial u}{\partial x}\right)^2$$

при начальном условии $u(x,0) = u^0(x)$, , где $f(x,t,u) \geqslant 0$, $a \geqslant 0$ константа, х одномерно. Продолжая применённые в [1] методы, в этой статье даётся явное конечно-разностное приближение для более общего уравнения (1.1) – (1.2).

Получены оценки устойчивости разностной схемы, они основываются на расщеплении оператора Au. Далее, с помощью двух теорем компактности доказывается сходимость метода.

MTA Számítástechnikai és Automatizálási Kutató Intézete, **Közlemények** 12. (1974)

Arató Mátyás:

A LINEÁRIS FILTRÁCIÓ VIZSGÁLATA DISZKRÉT GAUSS FOLYAMATOK ESETÉN

A lineáris filtráció Kálmán–Bucy egyenleteit vizsgáljuk abban a speciális esetben, amikor az ismeretlen idősor lineáris konstans együtthatós differencia egyenletnek tesz eleget s a megfigyelhető folyamat a nem megfigyelhetőből ugyancsak lineáris leképzéssel adódik, egy additiv zaj hozzáadásával. A tárgyalás ebben az esetben egészen elemi s nem kíván speciális valószínűségelméleti felkészülést.

Az (Ω, ϑ, P) valószínűségi mezőn egymással egyszerű sztochasztikus kapcsolatban lévő (Θ_n, η_n) valószínűségi változó sorozatokat vizsgálunk, ahol η_n a megfigyelhető Θ_n pedig a nem megfigyelhető komponens és Θ_n négyzetes középben legjobb becslését kívánjuk megadni. Kálmán és Bucy [1] mutatták meg, hogy egyszerű esetekben a megoldás a kiinduló egyenletekhez hasonló sztochasztikus differencia (illetve a folytonos esetben differenciál) egyenletet elégít ki, azaz az optimális megoldásra egy algoritmus adható.

Legyen a Θ_n $n = 1, 2, \ldots,$ sorozat Markov tipusú Gauss, mégpedig elégítse ki a

$$(1) \qquad \Theta_n = A_{n-1}\Theta_{n-1} + F_n\epsilon_n, \quad \Theta_0 = 0, \quad (E\Theta_n = E\epsilon_n = 0)$$

egyenletet, ahol ϵ_n egy független, egységnyi szórású, Gauss sorozat és ϵ_n független F_Θ^{n-1}-től is, ahol F_Θ^{n-1} a Θ változók által generált σ-algebra $F_\Theta^{n-1} = \sigma(\omega : \Theta_{n-1}(\omega), \ldots, \Theta_0(\omega))$. Feltessszük, hogy Θ_n közvetlenül nem megfigyelhető, csak az η_n sorozaton keresztül, ahol

$$(2) \qquad \eta_n = C_n\Theta_n + G_n w_n, \qquad n = 1, 2, \ldots,$$

ahol w_n olyan független, egységnyi szórású, Gauss sorozat mely – az egyszerűség kedvéért – az $\{\epsilon_n\}$ sorozattól is független. Az A_n, C_n, F_n, G_n nem azonosan 0 mátrixok csak az időtől függnek s függetlenek az egyenletekben szereplő valószínűségi változóktól (a legegyszerűbb esetben a folyamatok egydimenziósak).

Jelölje $F_\eta^n = \sigma\{\omega : \eta_n(\omega), \eta_{n-1}(\omega), \ldots, \eta_0(\omega)\}$ az η változók által generált σ-algebrák sorozatát. Bevezetjük a következő valószínűségi változókat

$$(3) \qquad \hat{\Theta}_n = E(\Theta_n \mid F_\eta^n), \qquad n = 1, 2, \ldots,$$

$$(4) \qquad e_n = \Theta_n - \hat{\Theta}_n, \quad d_n = Ee_n^2, \text{ és } e_n \text{ független } F_\eta^n\text{-től}, \quad n = 1, 2, \ldots,$$

$$(5) \qquad \xi_n^0 = \eta_n - E(\eta_n \mid F_\eta^{n-1}), \qquad n = 1, 2, \ldots$$

A feltételes várható érték definíciója alapján $\hat{\Theta}_n$ és ξ_n^0 is olyan Gauss eloszlású változók,

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melyek $\eta_1, \eta_2, \dots, \eta_n$ lineáris kombinációi, továbbá ξ_n^0 független F_η^{n-1}-től és így egy független sorozatot alkot. A definíciók felhasználásával adódik ξ_n^0 következő előállítása

$$\begin{aligned}(6) \qquad \xi_n^0 &= C_n\Theta_n + G_n w_n - E(C_n\Theta_n + G_n w_n \mid F_\eta^{n-1}) = \\ &= C_n\Theta_n + G_n w_n - C_n A_{n-1}\hat{\Theta}_{n-1} = C_n\Theta_n + G_n w_n - C_n A_{n-1}(\Theta_{n-1} - e_{n-1}) \\ &= C_n F_n \epsilon_n + G_n w_n + C_n A_{n-1} e_{n-1}.\end{aligned}$$

A $\xi_1^0, \xi_2^0, \dots, \xi_n^0$ változók az $\eta_1, \eta_2, \dots, \eta_n$ változók olyan lineáris kombinációi, melyből egyértelműen kölcsönösen meghatározhatók a ξ_i^0 és η_i változók. Ugyanis a leképezés mátrixa olyan háromszög mátrix, melynek fődiagonálisában 1-ek vannak. Ez azt jelenti, hogy a ξ_i^0 által generált σ-algebra sorozat megegyezik az η_i által generált σ-algebrák sorozatával: $F_{\xi 0}^n = F_\eta^n$, $n = 1, 2, \dots$

Tekintsük a következő valószínűségi változó sorozatot

$$\begin{aligned}(7) \qquad \xi_n^1 &= \hat{\Theta}_n - A_{n-1}\hat{\Theta}_{n-1} = (\hat{\Theta}_n - \Theta_n) + A_{n-1}(\Theta_{n-1} - \hat{\Theta}_{n-1}) + \\ &+ (\Theta_n - A_{n-1}\Theta_{n-1}) = -e_n + A_{n-1}e_{n-1} + F_n\epsilon_n, \qquad n = 1, 2, \dots,\end{aligned}$$

könnyen látható, hogy a (7)-ben értelmezett ξ_n^1 sorozat ugyancsak független Gauss sorozat. Ugyanis (7) jobboldalán minden változó független F_η^{n-1}-től és így mivel ξ_m^1 F_η^m mérhető, ξ_n^1 független ξ_m^1-től is, ha $m < n$.

Mivel ξ_n^1 F_η^n-mérhető és egyben $F_{\xi 0}^n$ mérhető, nemcsak az $\eta_1, \eta_2, \dots, \eta_n$ változók, hanem a $\xi_1^0, \dots, \xi_n^0$ változók lineáris függvénye is. Továbbá, ϵ_n és F_η^{n-1} (tehát $F_{\xi 0}^{n-1}$) függetlensége miatt

$$\begin{aligned}(8) \qquad \xi_n^1 &= \hat{\Theta}_n - A_{n-1}\hat{\Theta}_{n-1} = E(\Theta_n \mid F_\eta^n) - A_{n-1}E(\Theta_{n-1} \mid F_\eta^{n-1}) = \\ &= E(\Theta_n \mid F_{\xi 0}^n) - A_{n-1}E(\Theta_{n-1} \mid F_{\xi 0}^n) = E(\Theta_n - A_{n-1}\Theta_{n-1} \mid F_{\xi 0}^n) = \\ &= E(F_n\epsilon_n \mid F_{\xi 0}^n) = K_n\xi_n^0.\end{aligned}$$

Innen ξ_n^0-al való szorzással

$$(9) \qquad E(\xi_n^1\xi_n^0) = K_n E(\xi_n^0)^2.$$

A K_n együttható meghatározása a következőképpen történik. Egyrészt (6) alapján

$$\begin{aligned}(10) \qquad E(\xi_n^0\xi_n^0) &= E[C_nF_n\epsilon_n + G_n w_n + C_n A_{n-1}e_{n-1}]^2 = \\ &= (C_nF_n)^2 + G_n^2 + (C_nA_{n-1})^2 d_{n-1},\end{aligned}$$

ahol felhasználtuk ϵ_n és w_n függetlenségét, valamint ezek függetlenségét (4) és (1) alapján e_{n-1}-től. Továbbá (7) és (6) alapján

$$\begin{aligned}(11) \qquad E(\xi_n^1\xi_n^0) &= E(-e_n + A_{n-1}e_{n-1} + F_n\epsilon_n)(C_nF_n\epsilon_n + G_n w_n + \\ &+ C_nA_{n-1}e_{n-1}) = E(A_{n-1}e_{n-1} + F_n\epsilon_n)(C_nF_n\epsilon_n + G_n w_n + \\ &+ C_nA_{n-1}e_{n-1}) = C_n(F_n)^2 + A_{n-1}C_nA_{n-1}d_{n-1},\end{aligned}$$

felhasználva e_n és ξ_n^0 függetlenségét, valamint a már említett többi változó függetlenségét. (9)-ből (10) és (11) alapján

$$(12)\qquad K_n = \frac{E(\xi_n^1 \xi_n^0)}{E(\xi_n^0)^2} = [C_n(F_n)^2 + A_{n-1}C_nA_{n-1}d_{n-1}][(C_nF_n)^2 + G_n^2 + (C_nA_{n-1})^2d_{n-1}]^{-1}$$

A (8) összefüggésből a $\hat{\Theta}_n$ sorozatra a következő rekurziót kapjuk

$$(13)\qquad \hat{\Theta}_n = A_{n-1}\hat{\Theta}_{n-1} + K_n\zeta_n^0$$

ahol ζ_n^0 az (5) szerinti független sorozat.
A d_n szórásnégyzetek sorozatára egy differencia egyenlet írható fel.
Ugyanis $d_0 = 0$

$$d_n = E(\Theta_n - \hat{\Theta}_n)^2$$

továbbá

$$(14)\qquad d_n = E[(\Theta_n - \hat{\Theta}_n)\Theta_n - (\Theta_n - \hat{\Theta}_n)\hat{\Theta}_n] =$$
$$= E(\Theta_n)^2 - E(\Theta_n\hat{\Theta}_n) = E(\Theta_n)^2 - E(E(\Theta_n\hat{\Theta}_n \mid F_\eta^n)) = E(\Theta_n)^2 - E(\hat{\Theta}_n)^2 .$$

Viszont (1) alapján

$$E(\Theta_n)^2 = A_{n-1}E(\Theta_{n-1})^2A_{n-1} + (F_n)^2$$

és (13) alapján (10) felhasználásával

$$E(\hat{\Theta}_n)^2 = A_{n-1}E(\hat{\Theta}_{n-1})^2A_{n-1} + K_nE(\zeta_n^0)^2K_n .$$

Behelyettesítve (14)-be a két utolsó összefüggést és K_n (12) szerinti értékét

$$(15)\qquad d_n = A_{n-1}E(\Theta_{n-1})^2A_{n-1} + (F_n)^2 - [A_{n-1}E(\hat{\Theta}_{n-1})^2A_{n-1} +$$
$$+ \frac{[E(\zeta_n^1\zeta_n^0)]^2}{E(\zeta_n^0)^2}$$
$$= A_{n-1}d_{n-1}A_{n-1} + (F_n)^2 - \frac{[C_n(F_n)^2 + A_{n-1}C_nA_{n-1}d_{n-1}]^2}{(C_nF_n)^2 + G_n^2 + (C_nA_{n-1})^2d_{n-1}}$$

adódik.

A fenti tárgyalás akkor is igaz marad, ha Θ és η vektor folyamatok. A megfelelő képletek felírását az olvasó is elvégezheti a számítások végig követésével.

Ha az (1) és (2) lineáris egyenletek helyett a következő egyenletek teljesülnek

$$(16)\qquad \Theta_n = A(\Theta_{n-1}, \eta_{n-1}, n) + F_n\epsilon_n, \qquad \Theta_0 = 0,$$

és

$$(17) \qquad \eta_n = C(\Theta_{n-1}, \eta_{n-1}, n) + G_n w_n ,$$

ahol mint az előbbiekben ϵ_n és w_n független Gauss sorozatok, melyek egymástól is függetlenek a nemlineáris filtrációra jutunk. Ekkor a (Θ_n, η_n) pár egy nemlineáris sztochasztikus egyenletrendszer megoldása. Az előbbiekkel ellentétben ha $A(x, y, n)$ és $C(x, y, n)$ nem lineáris függvények Θ_n és η_n nem lesznek többé Gauss eloszlású változók.
Megmutatható, hogy a

$$\hat{\Theta}_\eta = E(\Theta_n \mid F^n_\eta)$$

sorozat ekkor is kielégít egy (7)-hez hasonló differencia egyenletet. A megfelelő, de időben folytonos eset tárgyalása megtalálható Lipcer és Sirjájev [2] összefoglaló dolgozatában. A (16) és (17) egyenletekkel leírt diszkrét folyamat nemlineáris filtrációjának elemi tárgyalásával egy későbbi dolgozatban foglalkozom.

Irodalom

[1] Kalman, R. E., Bucy, R. C., ”New results in linear filtering and prediction theory” Trans. ASME Journ. Basic Engr. 83 D (1961) 95-108.

[2] Липцер, Р.Ш., Ширяев, А.Н., "Нелинейная фильтрация диффузионных процессов" Труды МИАН I04 (I968) I35-I80.

Summary

The linear filtering problem of discrete Gaussian processes

If the process (Θ_n, η_n), $n = 1, 2, \ldots$ is given by equations (1) and (2) the filtering problem of Θ_n, when the observed process is η_n, may be solved by equations (13) and (12). An elementary proff of this statement is given.

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Линейная фильтрация дискретных гауссовских процессов

Если процесс (Θ_n, η_n); $n = 1, 2, \ldots$; задан уравнениями (1) и (2), где процесс η_n является "наблюдаемым" а Θ_n ненаблюдаемым, то линейная фильтрация процесса Θ_n решается с помощью уравнений (12) и (13). В статье дается элементарное доказательство этого утверждения.

MTA Számítástechnikai és Automatizálási Kutató Intézete, **Közlemények** 12. (1974)

ON THE MAXIMUM LIKELIHOOD ESTIMATION OF THE NUMBER OF SERVERS FOR THE M/M/m QUEUEING SYSTEM

By Jacob Eshak Samaan

INTRODUCTION

Many authors have investigated the statistical inferences for processes of queueing theory.

Maximum Likelihood estimators of the arrival rate and service rate for single and many servers queueing systems are also investigated.

However, it may occur that the arrival and service rates of a many server Markovian queueing system are known, and estimator for the number of servers is required. In this case we have the problem of estimating a discrete parameter which takes on integral values only.

In this paper we discuss the maximum Likelihood method for estimating the number m of servers assuming that we have a complete observations about the arrival and departure times of the customers. And we give also some numerical examples and illustrations.

To make the paper self-contained, we give at the end of it an appendix on the calculas of finite differences which is used to find the maximum of the likelihood function. Programs used to simulate the system are also presented in appendix 2.

NOTATIONS

Assuming that we observe the system for a period of time of length T, let us use the following notations:

$i \equiv$ the state of the system, i.e. the number of customers in the system (number of busy servers + length of waiting line),

$a_i \equiv$ the number of transitions from the state i to the state $i+1$ during the time of observations,

$b_i \equiv$ the total number of transitions from the state i to the state $i-1$ during the time of observations,

$T_i \equiv$ the total time in which the system stays in the state i, upto total observations time T,

$A \equiv \sum_{i=0}^{\infty} a_i \equiv$ the total number of positive jumps, in the sample,

$B \equiv \sum_{i=0}^{\infty} b_i \equiv$ the total number of negative jumps, in the sample.

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LIKELIHOOD FUNCTION OF THE SYSTEM M/M/m

Consider the case of queueing system in which the service centre possesses m servers in parallel, with common waiting line and are independent.

The arrival process is poisson with known parameter λ and the duration of service for each server is exponentially distributed with known parameter μ. If a customer arrives when all the servers are busy, he joins the waiting line. This system is described by a Markov process with state space $X = \{0, 1, 2, \ldots\}$, where the state represents the number of busy servers plus the length of the waiting line.

If we assume that the utilization factor $\rho = \lambda/\mu m$ is less than one, then the likelihood function is given by:

$$L_T(\Theta) = A \ln \lambda + B \ln \mu - \lambda T + C - \mu S$$

where:

$$C = \sum_{i=1}^{m} b_i \ln i + \sum_{i=m+1}^{\infty} b_i \ln m\,,$$

$$S = \sum_{i=1}^{m} iT_i + \sum_{i=m+1}^{\infty} m\,T_i\,.$$

(See for example Huybrechts [1].)
Note that the values A and B does not depend on the number m of servers.

MAXIMUM LIKELIHOOD ESTIMATOR FOR m

To find the maximum Likelihood estimation for the number of servers, we have to maximize the Likelihood function $L_T(\Theta)$, i.e. to find the value $\hat{m}$ which maximizes the function.

Since the first three terms of the Likelihood function does not depend on m, we have to find $\hat{m}$ which maximizes the function:

$$F(m) = C - \mu S =$$

$$= \left(\sum_{i=1}^{m} b_i \ln i + \sum_{i=m+1}^{\infty} b_i \ln m\right) - \mu\left(\sum_{i=1}^{m} iT_i + \sum_{i=m+1}^{\infty} mT_i\right) =$$

$$= \sum_{i=1}^{m} (b_i \ln i - i\mu T_i) + \sum_{i=m+1}^{\infty} (b_i \ln m - m\mu T_i)\,.$$

For the maximum value of $F(m)$ we require the first value of m which satisfy the relationship:

$$\Delta F(m) < 0 < \Delta F(m-1)\,,$$

where $\Delta F(m)$ is the first difference of the function $F(m)$, as defined in appendix (1). To obtain $\Delta F(m)$ we refer to the appendix. From the criterion given there we see that we can difference under the summation sign provided it is true that:

$$b_i \ln i - i\mu T_i = b_i \ln(m+1) - (m+1)\mu T_i\,,$$

when i has the value $(m+1)$. Since both sides have the values of:

$$b_{m+1} \ln(m+1) - (m+1)\mu T_{m+1}\,,$$

the requirement is satisfied and we may difference under the summation sign to get:

$$\Delta F(m) = \sum_{i=m+1}^{\infty} \left(b_i \ln \frac{m+1}{m} - \mu T_i\right)$$

or

$$\Delta F(m) = \sum_{i=m+1}^{\infty} (Kb_i - \mu T_i)$$

where

$$K = \ln \frac{m+1}{m}$$

is a constant.
Thus the estimator $\hat{m}$ of m is the first value of m satisfying:

$$\Delta F(m) < 0 < \Delta F(m-1)$$

i.e.

$$\sum_{i=m+1}^{\infty} \left(b_i \ln \frac{m+1}{m} - \mu T_i\right) < 0 < \sum_{i=m}^{\infty} \left(b_i \ln \frac{m}{m-1} - \mu T_i\right).$$

Or:

$$\sum_{i=m+1}^{\infty} b_i \ln \frac{m+1}{m} < \mu \sum_{i=m+1}^{\infty} T_i < \sum_{i=m}^{\infty} b_i \ln \frac{m}{m-1} - \mu T_m\,.$$

So $\hat{m}$ is the first value of m which satisfy:

$$\sum_{i=m+1}^{\infty} (Kb_i - \mu T_i) < 0\,, \qquad K = \ln \frac{m+1}{m}\,.$$

STRUCTURE OF AN ALGORITHM FOR SIMULATING THE M/M/m QUEUEING SYSTEM

For the purpose of numerical illustrations of the above described method for estimating the number of servers, we need an algorithm to simulate the system and generate the arrays T'_iS, a'_iS and b'_iS in addition to the number L which represents the maximum number of customers presented simultaneously in the system. An algorithm for simulating any general many servers system may be given, but for our special case of Markovian system, there is, a simple algorithm.

Figure 1 shows a detailed logical block diagram of this algorithm.

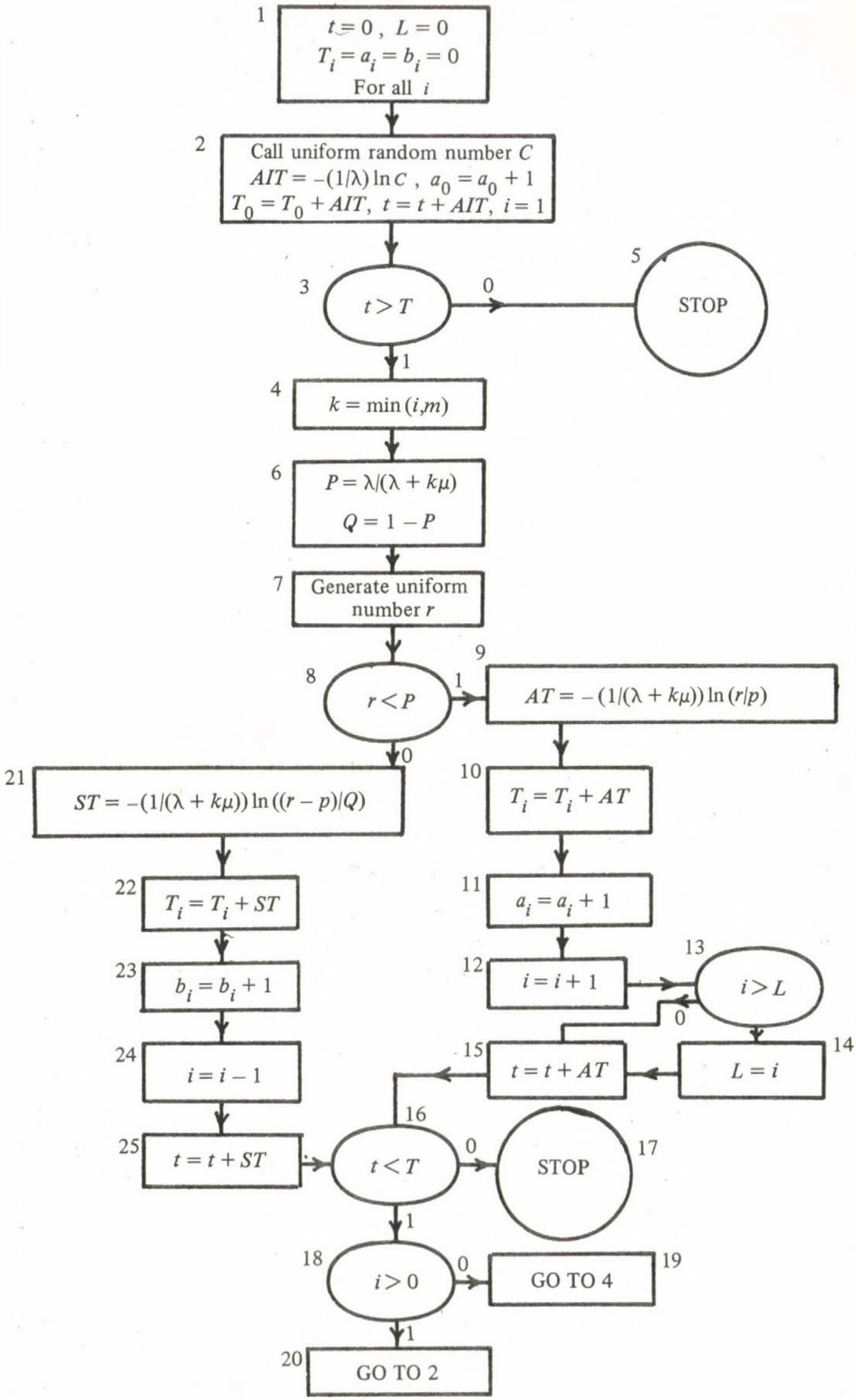
Logical operators are shown with circles. If condition tested by a given logical operator is satisfied, the arrow denoting the direction of control is qualified by the index 1, in the opposite case by index 0.

In the diagram the following new notations are also used:

t	$\equiv$	denote the cumulative observation time upto observing the customer under consideration,
AIT	$\equiv$	denote the length of an idle interval,
AT, ST	$\equiv$	denote the length of time interval, the system spends in a certain state given that the next transition will be an arrival or departure respectively.

Note that AIT is generated as an exponential random variables with parameter λ, and both AT and ST are generated as an exponential random variables, with parameter $\lambda + k\mu$.

A complete program written in FORTRAN language for estimating the number of servers and using this algorithm as a subprogram is presented in appendix (2).


1
$t = 0$, $L = 0$
$T_i = a_i = b_i = 0$
For all i
2
Call uniform random number C
$AIT = -(1/\lambda) \ln C$, $a_0 = a_0 + 1$
$T_0 = T_0 + AIT$, $t = t + AIT$, $i = 1$
3
$t > T$
0
5
STOP
1
4
$k = \min(i,m)$
6
$P = \lambda/(\lambda + k\mu)$
$Q = 1 - P$
7
Generate uniform number r
8
$r < P$
1
9
$AT = -(1/(\lambda + k\mu)) \ln (r/p)$
0
21
$ST = -(1/(\lambda + k\mu)) \ln ((r - p)/Q)$
10
$T_i = T_i + AT$
11
$a_i = a_i + 1$
12
$i = i + 1$
13
$i > L$
0
14
$L = i$
15
$t = t + AT$
22
$T_i = T_i + ST$
23
$b_i = b_i + 1$
24
$i = i - 1$
25
$t = t + ST$
16
$t < T$
0
17
STOP
1
18
$i > 0$
0
19
GO TO 4
1
20
GO TO 2

TABLES 1 AND 2 GIVE TWO SAMPLES GENERATED BY THIS ALGORITHM FOR T=1000 , M=3. THE THEORETICAL STATIONARY PROBABILITIES ARE PRESENTED IN THE LAST COLUMN.

L= 7

I	T(I)	A(I)	B(I)	P(I)
0	227.052146	119	0	0.278607
1	361.872123	191	118	0.348259
2	251.647517	118	191	0.217662
3	98.215665	54	118	0.090692
4	48.336677	16	54	0.037788
5	11.028369	4	16	0.015745
6	[illegible].946499	1	4	0.006561
7	2.124036	0	1	0.002734

TABLE 1

LANDA=0.5 , MEW=0.4 , ROW=0.417

L=32

I	T(I)	A(I)	B(I)	P(I)
0	63.406063	59	0	0.044944
1	139.839975	127	58	0.112360
2	151.161000	152	126	0.140449
3	117.441189	113	151	0.117041
4	96.509010	105	112	0.097534
5	72.332199	69	104	0.081279
6	54.870621	45	68	0.067732
7	34.380235	35	44	0.056443
8	31.628616	36	35	0.047036
9	26.329489	22	36	0.039197
10	16.761720	25	22	0.032664
11	29.767141	29	25	0.027220
12	22.591297	17	29	0.022683
13	25.079059	19	17	0.018903
14	12.612390	11	19	0.015752
15	7.603973	4	11	0.013127
16	2.956685	3	4	0.010939
17	4.976889	5	3	0.009116
18	4.114965	6	5	0.007597
19	9.459137	7	6	0.006331
20	10.537473	11	7	0.005275
21	8.024612	10	11	0.004396
22	14.361141	13	10	0.003663
23	9.423197	12	13	0.003053
24	7.600012	6	12	0.002544
25	4.956691	5	6	0.002120
26	7.994598	8	5	0.001767
27	4.890794	9	8	0.001472
28	3.733509	3	9	0.001227
29	2.138120	1	3	0.001022
30	1.806712	2	1	0.000852
31	1.483525	1	2	0.000710
32	0.104564	0	1	0.000592

TABLE 2

LANDA=1.0 , MEW=0.4 , ROW=0.833

SOME NUMERICAL EXAMPLES AND ILLUSTRATIONS

Using the above algorithm we get 100 samples by CDC 3300 Computer. Estimates of the number of servers using the maximum Likelihood method of estimation are also founded.

The following tables summarize some of these results for different values of the parameters λ, μ, m and T.

Let N_i denote the number of times for which the estimated value of the number of servers is equal to i out from 100 samples, and let N^* denote the number of times we get the true value of the number of servers out of the hundred samples. For example, if we have $m = 3$, then $N^* = N_3$.

Table 3 gives the results for fixed values of T and m, and different values of λ and μ.

We have $T = 1000$, $m = 3$.

λ	1.0	1.0	1.0	0.5	0.5
μ	0.4	0.5	0.75	0.4	0.6
ρ	5/6	2/3	4/9	5/12	5/18
N_2	0	0	0	0	2
$N_3 = N^*$	100	100	95	93	73
N_4	0	0	5	6	18
N_5	0	0	0	0	7
N_6	0	0	0	1	0

Table 3.

Now, ıı we fix ρ and T while give different values for m, table 4 shows the results for $T = 1000$, $\rho = 5/12$, we have:

λ	0.5	0.5	0.5	0.5	0.5
μ	0.6	0.4	0.3	0.24	0.2
m	2	3	4	5	6
N_2	**98**	0	0	0	0
N_3	2	**90**	6	0	0
N_4	0	5	**64**	18	2
N_5	0	2	18	**46**	15
N_6	0	1	6	17	**42**
N_7	0	2	3	12	22
N_8	0	0	2	5	10
N_9	0	0	1	2	7
N_{10}	0	0	0	0	1
N_{11}	0	0	0	0	1

Table 4.

In this table the number N^* defined above is heavy written in each case.

Finally, table 5 illustrates the results when we fix the values of ρ and for different values of T, we have:

$\lambda = 0.5$, $\mu = 0.4$, $m = 3$ i.e. $\rho = 5/12$.

T	50	100	250	500	1000	10 000
N_1	2	0	0	0	0	0
N_2	23	15	9	3	0	0
$N_3 = N^*$	45	41	65	76	85	100
N_4	20	30	14	13	14	0
N_5	10	10	11	4	1	0
N_6	0	3	1	4	0	0
N_7	0	1	0	0	0	0

Table 5.

From these tables we note that the procedure works with large accuracy for the large values of ρ. It is clear from table 3. that for the values of ρ lies in the interval $0.5 \leqslant \rho < 1$, the results are very satisfactory.

From table 4. we see that the procedure works better when the number of servers is small.

In fact when ρ increases while m remains fixed, the expected queue length increases and (as it is clear from tables 1. and 2.), L also increases. Let P_i^* denote the stationary probabilities of finding the system in the state i. It is given by the following formulas:

$$P_i^* = \frac{1}{i!}\left(\frac{\lambda}{\mu}\right)^i P_0^* \qquad i < m\,,$$

$$= \frac{1}{m!m^{i-m}}\left(\frac{\lambda}{\mu}\right)^i P_0^* \qquad i \geqslant m\,,$$

where

$$P_0^* = \left[\sum_{i=0}^{m-1} \frac{1}{i!}\left(\frac{\lambda}{\mu}\right) + \frac{1}{m!}\left(\frac{\lambda}{\mu}\right)^m \cdot \frac{m\mu}{m\mu - \lambda}\right]^{-1}.$$

Thus, for $i \geqslant m$ we have

$$P_i^* = \rho P_{i-1}^*$$

and when ρ has large value near to one, the probabilities P_i^*, which appear in the formula representing $\Delta F(m)$, decreases slowly. So the summation in this formula contains a large number of terms. This may be the reason of obtaining better estimators for the large values of ρ.

On the other hand, when we increase m while keep ρ fixed, then L will be nearly the same, but the lower bound of the summation representing $\Delta F(m)$ increases.

This means that our estimation bases on a small number of states, and in the same time the probabilities P_i^* of finding the system in these states, which have indices $i > m$ are small when m is large and thus the relative error in its simulated values is large.

More precisely, we may say that, our method has no meaning in the case of very large m. Because in this case we have $K = \ln\frac{m+1}{m} \cong 0$ and all the b_i's and T_i's are very small for $i \geqslant m$.

This may explain the bad results we got for the large values of m.

Also, from table 5, we see that the procedure works with a very high accuracy for the large values of T and gives the true value of the number of servers all the times when $T = 10\,000$.

In fact, for the large values of T, the random variables $b_i/\lambda TP_{i-1}^*$, T_i/TP_i^* and also $a_i/\lambda TP_i^*$ stochastically converge to one. Thus we have the following approximations for the sequences b_i's, T_i's and a_i's:

$$b_i \cong \lambda TP_{i-1}^*, \quad T_i \cong TP_i^*, \quad a_i \cong \lambda TP_i^*.$$

Tables 6. and 7. show, how the above random variables approache the theoretical probabilities, when T increases.

Now if we cousider the function:

$$F(k) = \sum_{i=1}^{k} (b_i \ln i - i\mu T_i) + \sum_{i=k+1}^{\infty} (b_i \ln k - k\mu T_i),$$

then we can see that after this approximations carried out, this function has only a unique maximum, and that, this maximum occur at $k = m$, which is the true value of the number of servers. In this case:

$$\Delta F(k) = \mu TP_0^* \left[\left(\frac{\lambda C}{\mu} \sum_{j=k}^{m-1} \frac{1}{j!}\left(\frac{\lambda}{\mu}\right)^j - \sum_{i=k+1}^{m} \frac{1}{i!}\left(\frac{\lambda}{\mu}\right)^i \right) \right.$$
$$\left. + \left(\frac{\lambda C}{\mu} \sum_{j=m}^{\infty} \frac{1}{m!m^{j-m}} \left(\frac{\lambda}{\mu}\right)^j - \sum_{i=m+1}^{\infty} \frac{1}{m!m^{i-m}}\left(\frac{\lambda}{\mu}\right)^i \right) \right]$$

PROBABILITIES CALCULATED ON THE BASE OF B(I).

I	T=50	T=100	T=250	T=500	T=1000	T=10000	P(I)
0	0.480000	0.280000	0.272000	0.312000	0.282000	0.275000	0.278607
1	0.640000	0.260000	0.328000	0.280000	0.350000	0.343000	0.348259
2	0.120000	0.030000	0.272000	0.176000	0.230000	0.215800	0.217662
3	0.040000	0.040000	0.168000	0.068000	0.106000	0.089400	0.090692
4	0.000000	0.000000	0.056000	0.032000	0.032000	0.041600	0.037788
5	0.000000	0.020000	0.008000	0.020000	0.014000	0.019200	0.015745
6	0.000000	0.000000	0.000000	0.000000	0.002000	0.009400	0.006561
7	0.000000	0.020000	0.000000	0.000000	0.000000	0.004000	0.002734
8	0.000000	0.020000	0.000000	0.000000	0.004000	0.001800	0.001139
9	0.000000	0.000000	0.000000	0.000000	0.000000	0.000600	0.000475
10	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000198
11	0.000000	0.000000	0.000000	0.000000	0.000000	0.000200	0.000082
12	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000034

TABLE 6

LANDA=0.5 , MEW=0.4 , M=3

THE VALUES OF $b_{i+1}/\lambda T$ FOR DIFFERENT VALUES OF T.

IN THE LAST COLUMN THE THEORETICAL STATIONARY

PROBABILITIES ARE PRESENTED.

PROBABILITIES CALCULATED ON THE BASE OF T(I).

I	T=50	T=100	T=250	T=500	T=1000	T=10000	P(I)
0	0.319226	0.330009	0.230937	0.362305	0.277842	0.279843	0.278607
1	0.419161	0.445298	0.283561	0.331468	0.346696	0.344387	0.348259
2	0.229242	0.205329	0.258789	0.166570	0.225258	0.215487	0.217662
3	0.028611	0.023363	0.145207	0.077662	0.097839	0.085627	0.090692
4	0.016434	0.005873	0.051598	0.035899	0.032148	0.038479	0.037788
5	0.000000	0.006313	0.029813	0.024589	0.015347	0.020155	0.015745
6	0.000000	0.001901	0.000593	0.005041	0.003692	0.007786	0.006561
7	0.000000	0.013006	0.000590	0.000516	0.001413	0.005549	0.002734
8	0.000000	0.013943	0.000000	0.000000	0.004194	0.001447	0.001139
9	0.000000	0.010774	0.000000	0.000000	0.000630	0.001021	0.000475
10	0.000000	0.000000	0.000000	0.000000	0.000000	0.000246	0.000193
11	0.000000	0.000000	0.000000	0.000000	0.000000	0.000154	0.000082
12	0.000000	0.000000	0.000000	0.000000	0.000000	0.000075	0.000034

TABLE 7

LANDA=0.5 , MEW=0.4 , M=3

THE VALUES OF T_i/T FOR DIFFERENT VALUES OF T.

IN THE LAST COLUMN THE THEORETICAL STATIONARY

PROBABILITIES ARE PRESENTED.

where $C = \ln \frac{k+1}{k}$.

i.e.

$$\Delta F(k) = \mu T P_0^* \left[\left(\frac{\lambda C}{\mu} \sum_{j=k}^{m-1} \frac{1}{j!} \left(\frac{\lambda}{\mu}\right)^j - \frac{\lambda}{\mu} \sum_{j=k}^{m-1} \frac{1}{(j+1)!} \left(\frac{\lambda}{\mu}\right)^j \right) \right.$$

$$\left. + \frac{1}{m!m} \left(\frac{\lambda}{\mu}\right)^{m+1} \cdot \frac{1}{1-\rho} (mC - 1) \right] =$$

$$= \lambda T P_0^* \left[\sum_{j=k}^{m-1} \left(\frac{\lambda}{\mu}\right)^j \left(\frac{C}{j!} - \frac{1}{(j+1)!} \right) + \frac{(mC-1)}{m!m(1-\rho)} \left(\frac{\lambda}{\mu}\right)^m \right].$$

To prove that a maximum of the function $F(k)$ occurs at $k = m$, we have to prove that m is the first value of k for which $\Delta F(k) < 0$.

That is, we have to prove that:

$$\Delta F(k) > 0 \quad \text{for all} \quad k < m,$$

and

$$\Delta F(m) < 0.$$

For the proof of the first part, we shall prove that the function $\Delta F(k)$ is a decreasing function for $k < m$, and

$$\Delta F(m-1) > 0.$$

Let:

$$R(k) = \frac{\Delta F(k)}{\lambda T P_0^*} = \sum_{j=k}^{m-1} \left(\frac{\lambda}{\mu}\right)^j \left[\frac{C}{j!} - \frac{1}{(j+1)!} \right] + \frac{(mC-1)}{m!m(1-\rho)} \left(\frac{\lambda}{\mu}\right)^m.$$

To see that $\Delta F(k)$ is a decreasing function, we shall prove that:

$$\Delta R(k) < 0 \quad \text{for all} \quad k < m,$$

we have

$$\Delta R(k) = \left\{ \sum_{j=k+1}^{m-1} \left(\frac{\lambda}{\mu}\right)^j \left[\frac{1}{j!} \ln \frac{k+2}{k+1} - \frac{1}{(j+1)!}\right] + \frac{\left(\frac{\lambda}{\mu}\right)^m}{m!m(1-\rho)} \left[m \ln \frac{k+2}{k+1} - 1\right]\right\}$$

$$- \left\{ \sum_{j=k}^{m-1} \left(\frac{\lambda}{\mu}\right)^j \left[\frac{1}{j!} \ln \frac{k+1}{k} - \frac{1}{(j+1)!}\right] - \frac{\left(\frac{\lambda}{\mu}\right)^m}{m!m(1-\rho)} \left[m \ln \frac{k+1}{k} - 1\right]\right\} =$$

$$= \frac{1}{m!(1-\rho)} \left(\frac{\lambda}{\mu}\right)^m \ln \frac{k(k+2)}{(k+1)^2} - \left(\frac{\lambda}{\mu}\right)^k \left[\frac{1}{k!} \ln \frac{k+1}{k} - \frac{1}{(k+1)!}\right]$$

$$+ \sum_{j=k+1}^{m-1} \left(\frac{\lambda}{\mu}\right)^j \left[\left(\frac{1}{j!} \ln \frac{k+2}{k+1} - \frac{1}{(j+1)!}\right) - \left(\frac{1}{j!} \ln \frac{k+1}{k} - \frac{1}{(j+1)!}\right)\right]$$

i.e.

$$\Delta R(k) = \frac{1}{m!(1-\rho)} \left(\frac{\lambda}{\mu}\right)^m \ln \frac{k(k+2)}{(k+1)^2} + \sum_{j=k+1}^{m-1} \left(\frac{\lambda}{\mu}\right)^j \ln \frac{k(k+2)}{(k+1)^2}$$

$$- \frac{1}{(k+1)!} \left(\frac{\lambda}{\mu}\right)^k \left[(k+1) \ln \frac{k+1}{k} - 1\right]$$

Since for every k we have:

$$\frac{k(k+2)}{(k+1)^2} < 1 ,$$

then the first two terms are always negative for each k and it remains only to prove that:

$$(k+1) \ln \frac{k+1}{k} - 1 > 0 .$$

But:

$$(k+1) \ln \frac{k+1}{k} - 1 = k \ln(1 + \tfrac{1}{k}) + \ln(1 + \tfrac{1}{k}) - 1$$

$$= k\left[\frac{1}{k} - \frac{1}{2k^2} + \frac{1}{3k^3} - \ldots\right] - 1$$

$$+ \left[\frac{1}{k} - \frac{1}{2k^2} + \frac{1}{3k^3} - \ldots\right]$$

$$= k(1 - \tfrac{1}{2}) - \frac{1}{k^2}\left(\frac{1}{2} - \frac{1}{3}\right) + \frac{1}{k^3}\left(\frac{1}{3} - \frac{1}{4}\right) \ldots$$

Since $k \geqslant 1$, then Leibinz theorem shows that this series converges to positive quantity as required.

This completes the proof that $\Delta F(k)$ is a decreasing function of k when $k < m$.

Now

$$\Delta F(m-1) = \sum_{i=m}^{\infty} (\lambda T P_{i-1}^* \ln \frac{m}{m-1} - \mu T P_i^*) =$$

$$= \frac{\mu T P_0^*}{m!(1-\rho)} \left(\frac{\lambda}{\mu}\right)^m \left[m \ln \frac{m}{m-1} - 1\right] .$$

But, as before, we have

$$m \ln \frac{m}{m-1} - 1 > 0 .$$

This means that

$$\Delta F(m-1) > 0 .$$

Let us now consider the value of:

$$\Delta F(m) = \sum_{i=m+1}^{\infty} (\lambda T P_{i-1}^* \ln \frac{m+1}{m} - \mu T P_i^*) =$$

$$= \frac{\mu T P_0^*}{m!(1-\rho)} \left(\frac{\lambda}{\mu}\right)^m \left[m \ln \frac{m+1}{m} - 1\right]$$

But:

$$m \ln \frac{m+1}{m} - 1 = m \ln (1 + \frac{1}{m}) - 1$$

$$= m \left[\frac{1}{m} - \frac{1}{2m^2} + \frac{1}{3m^3} - \ldots\right] - 1$$

$$= - \frac{1}{2m} + \frac{1}{3m^2} - \frac{1}{4m^3} + \ldots \quad < 0$$

This proves that m is the first value of k which satisfy the inequality

$$\Delta F(k) < 0 ,$$

and consequently, a maximum of the function $F(k)$ occurs at $k = m$ as required.

We continue to prove that this maximum is unique. To see this we prove that $F(k)$ is a decreasing function for $k > m$, i.e.

$$\Delta F(k) < 0 \quad \text{for} \quad k > m .$$

We have:

$$\Delta F(k) = \sum_{i=k+1}^{\infty} (\lambda T P_{i-1}^{*} \ln \frac{k+1}{k} - \mu T P_i^{*}) =$$

$$= \frac{\mu T P_0^{*}}{m!m(1-\rho)} \left(\frac{\lambda}{\mu}\right)^{m+1} [m \ln \frac{k+1}{k} - 1] , \qquad k > m$$

But:

$$m \ln \frac{k+1}{k} - 1 = m \ln (1 + \frac{1}{k}) - 1$$

$$= (\frac{m}{k} - 1) - \frac{1}{2k^2} + \frac{1}{3k^3} \ldots < 0 \quad \text{since} \quad \frac{m}{k} < 1 .$$

Thus

$$\Delta F(k) < 0 \quad \text{for all} \quad k > m .$$

This completes the proof that the absolute maximum of $F(k)$ occurs at $k = m$.

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APPENDIX (1)

FINITE DIFFERENCES

The use of the calculas of finite differences can lead to a simplification of notation in many numerical process. Many of the results are directly analogues to those of infinitismal calculas, and in fact, many formulas of the latter can be derived in a simple mannar from the corresponding formulas of the finite difference calculas.

Definition of the difference operator Δ

We shall be concerned with functions $f(n)$ defined only for integral values of the argument n. The first difference of $f(n)$ is denoted by $\Delta f(n)$ and is defined by the formula:

$$\Delta f(n) = f(n+1) - f(n) .$$

Second and subsequent differences of $f(n)$ are defined by:

$$\Delta^{r+1} f(n) = \Delta \{\Delta^r f(n)\} .$$

We make the conventions:

$$\Delta^0 \equiv \text{identify operator}, \quad \Delta' = \Delta$$

Conditions for a maximum of $f(n)$

The function $f(n)$ will have a "Local maximum" at n_0 provided both the following conditions are satisfied:

$$f(n_0 + 1) < f(n_0) , \quad \text{i.e.} \quad \Delta f(n_0) < 0 ,$$
$$f(n_0 - 1) < f(n_0) , \quad \text{i.e.} \quad \Delta f(n_0 - 1) > 0 .$$

Thus $f(n)$ will have a local maximum at n_0 if:

$$\Delta f(n_0) < 0 < \Delta f(n_0 - 1) .$$

The function $f(n)$ is said to have an "absolute maximum" at n_0 if $f(n_0) \geqslant f(n)$ for all n. Sufficient condition for $f(n)$ to have an absolute maximum at n_0 are that:

$$\Delta f(n_0) < 0 < \Delta f(n_0 - 1) ,$$

be satisfied and also that:

$$\Delta^2 f(n) \leqslant 0 \qquad \text{for all } n.$$

The conditions for $f(n)$ to assume a minimum at n_0 are analogues.

Differences under a summation sign

In many problems, we offen run into the necessity for computing the difference of some functions $F(m)$ of the form:

$$F(m) = \sum_{i=a(m)}^{b(m)}{}' f(i, m)$$

where the boundaries $a(m)$ and $b(m)$ of the region are increasing functions of m. In this case we have:

$$F(m) = \sum_{i=a(m)}^{b(m)}{}' \Delta f(i, m) + \sum_{i=b(m)+1}^{b(m+1)}{}' f(i, m+1) - \sum_{i=a(m)}^{a(m+1)-1}{}' f(i, m+1)\,.$$

However, if $f(i, m)$ has different functional forms in different sectors of the region of summation, such as:

$$(1) \qquad f(i, m) = \begin{cases} f_1(i, m) & \text{for } i \text{ in the interval } 0 \leqslant i \leqslant b(m) \\ f_2(i, m) & \text{for } i > b(m)\,, \end{cases}$$

and we wish to difference the function:

$$F(m) = \sum_{i=0}^{\infty}{}' f(i, m)\,,$$

then we have the following:

CRITERION

If

$$F(m) = \sum_{i=0}^{\infty}{}' f(i, m)$$

where $f(i, m)$ defined by (1), then:

$$\Delta F(m) = \sum_{i=0}^{\infty}{}' \Delta f(i, m)$$

provided that $f_1(i, m+1)$ equals $F_2(i, m+1)$ for all i in the interval $b(m) + 1 \leqslant i \leqslant b(m+1)$.

For the proof of this criterion, and for other details about finite differences, see for example Sasieni, Yaspan, and Friedman [2].

```
C
C
C
C                                        APPENDIX (2)
C
C
C
C       ****************************************************
C       *                                                  *
C       *   MAIN PROGRAM FOR ESTIMATING THE NUMBER         *
C       *   OF SERVERS FOR A MANY SERVER MARKOVIAN         *
C       *          QUEUEING SYSTEM                         *
C       *                                                  *
C       ****************************************************
C
C
      DIMENSION MT(100),FMAX(100)
     1,D(100),IA(100),IB(100)
C
      CALL START(83547.92051)
      READ 1,AR,SR,M,TT
    1 FORMAT(F5.2,2X,F5.2,2X,I3,2X,F8.2)
C
      N=100
      DO 9 IE=1,N
      CALL CALC(AR,SR,M,TT,LL,D,IA0,IA,IB)
      LT=1
    4 LZ=LT+1
      Z=LZ*(1.0/LT)
      C=ALOG(Z)
      I1=LZ
      TD=0.0
      TIB=0.0
      DO 2 I=I1,LL
      TD=TD+D(I)
      TIB=TIB+IB(I)
    2 CONTINUE
      SUM=C*TIB-SR*TD
      IF(SUM.GT.0.0)       GO TO 3
      MT(IE)=LT
      FMAX(IE)=SUM
      GO TO 9
    3 LT=LT+1
      GO TO 4
    9 CONTINUE
C
      PRINT 20
   20 FORMAT(1H1)
      DO 23 L6=1,N
   23 PRINT 14, MT(L6),FMAX(L6)
   14 FORMAT(10X,I3,5X,F11.4)
C
      END
```

```
C
C
C
C     ..............................................
C
C     SUBROUTINE  CALC
C
C     PURPOSE
C        SIMULATING THE MANY SERVER MARKOVIAN
C        QUEUEING SYSTEM TO PRODUE THE NUMBER
C        OF TRANSITIONS FROM EACH STATE UPWARDS
C        AND DOWNWARDS,AND ALSO THE DURATION
C        OF THE SYSTEM IN EACH STATE UPTO A
C        PREFIXED TIME INTERVAL OF LENGTH  T
C
C     USAGE
C        CALL  CALC(AR,SR,M,TT,LL,D,IAP,IA,IB)
C
C
C     DESCRIPTION OF PARAMETERS:
C        AR    -THE ARRIVAL RATE OF CUSTOMERS.
C        SR    -THE SERVICE RATE OF CUSTOMERS.
C        M     -THE NUMBER OF SERVERS IN THE
C               SYSTEM.
C        TT    -THE TOTAL TIME OF OBSERVATIONS.
C        LL    -OUTPUT VARIABLE REPRESENTS THE
C               MAXIMUM NUMBER OF CUSTOMERS
C               PRESENTED SIMULTANEOUSLY IN THE
C               SYSTEM.
C        D     -OUTPUT VECTOR OF LENGTH LL+1
C               CONTAINING THE DURATION OF THE
C               SYSTEM IN EACH STATE
C        IAP   -AN OUTFUT VARIABLE GIVES THE
C               TOTAL NUMBER OF TRANSITIONS FROM
C               THE STATE ZERO UPWARDS DURING THE
C               TIME OF OBSERVATION.
C        IA    -AN OUTPUT VECTOR OF LENGTH LL
C               GIVES THE NUMBER OF TRANSITIONS
C               FROM ANY STATE (OTHER THAN THE
C               ZERO STATE) TO THE NEXT STATE
C               UPWARD IN THE  SAMPLE.
C        IB    -OUTPUT VECTOR OF LENGTH LL
C               REPRESENTS THE TOTAL NUMBER OF
C               TRANSISTIONS DOWNWARDS FROM ANY
C               STATE,DIFFERENT THAN THE ZERO
C               STATE,IN THE SAMPLE.
C               NOTE THAT THE NUMBER OF TRANSITIONS
C               FROM THE ZERO STATE DOWNWARDS IS
C               ALWAYS EQUAL TO ZERO
C
C     SUBROUTINES REQUIRED
C        RANDU
C
C     **********************************************
```

```
C
C
C
      SUBROUTINE CALC(AR,SR,M,TT,LL,D,IA0,IA,IB)
C
      DIMENSION D(1),IA(1),IB(1)
C
      T=0.0
      IA0=0
      IB0=0
      DS0=0.0
      MAXNCS=0
      LL=MAXNCS
      DO 19 LI=1,50
      D(LI)=0.0
      IA(LI)=0
      IB(LI)=0
   19 CONTINUE
    4 CALL RANDU(C)
      AIT=-(1/AR)*ALOG(C)
      DS0=DS0+AIT
      IA0=IA0+1
      I=1
      T=T+AIT
      GO TO 3
    5 IF(I.GT.M) GO TO 6
      K=I
      GO TO 7
    6 K=M
    7 PP=AR/(AR+K*SR)
      Q=1-PP
      CALL RANDU(R)
      IF(R.LT.PP) GO TO 8
      ST=-(ALOG((R-PP)/Q))/(AR+K*SR)
      D(I)=D(I)+ST
      IB(I)=IB(I)+1
      I=I-1
      T=T+ST
      GO TO 3
    8 AT=-(ALOG(R/PP))/(AR+K*SR)
      D(I)=D(I)+AT
      IA(I)=IA(I)+1
      I=I+1
      IF(I.GT.LL)     MAXNCS=I
      LL=MAXNCS
      T=T+AT
    3 IF(T.GE.TT)          GO TO 9
      IF(I.EQ.0) GO TO 4
      GO TO 5
    9 CONTINUE
      RETURN
      END
```

References

[1] Huybrechts, S., "Inference statistique dans les procès de Markov – Applications dans les problèmes de Files d'attente" Queuing theory: Recent developments and applications (A Conference under the aegis of NATO Science Commetee), 127 (1965) 127-142.

[2] Sasieni, M., Yaspan, A., Friedman, L., Operational research . . . methods and problems willey (New York, 1959) 294-303.

Summary

The queueing system M/M/m with unknown number m of servers is considered. Maximum Likelihood estimate for m is founded and uniqueness of the solution of the Likelihood function is proved when the time t of observation tends to infinity. Simulation technique is used to investigate the estimates proposed.

Резюме

Рассматривается система обслуживания M/M/m с неизвестным числом каналов. Найдена оценка максимального правдоподобия для m. Доказана единственность решения уравнения правдоподобия, когда время наблюдения неограниченно возрастает. Свойства оценки изучены при помощи метода Монте-Карло.

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A FIRST PASSAGE PROBLEM FOR AN M/M/1 QUEUE

By Ahmed S. Mashour

INTRODUCTION

Consider the queue size process $\{Q(t),\ 0 \leq t \leq T\}$ associated with the queueing system M/M/1. Let $L(t)$ be a given integral valued non-increasing step function with $L(0) > 0$, $L(T) = 0$. One purpose of this paper is to investigate the random variable $\tau = \inf\limits_{0<U<T} \{u : Q(u) \geq L(u)\}$ and its characteristics. An algorithm is presented in Sec.1 in order to obtain an explicit formula for the distribution function of τ.

In Sec.2 it is shown that such a problem arise when we deal with an M/M/1 system which offers service for the arriving customers during a finite interval $(0,T)$. Every served customer provides a revenue $r > 0$. After the closing time T, no new customers are admitted and the present customers, if any, are to be served in an overtime, at a runing cost C/unit time. The system has to be operated in order to make the expected net revenue as high as possible. The influence of the overtime costs on the net revenue, necessitates choosing a policy to control the input process.

A rejection time policy, closing the input earlier than T, is considered. A deterministic rejection time has been discussed in [2]. The random variable τ introduced represents a stopping time the optimal choice of which is discussed in sec.2.

A formula for the expected net revenue associated with a policy $L(t)$ is given in Sec.3. Then the deterministic and the described random rejection policies are compared.

1. THE DISTRIBUTION OF τ

We are concerned with an M/M/1 queueing system where the customers arrive in a Poisson stream at mean rate λ and the service times are independently, identically and exponentially distributed with mean $1/\mu$. We assume that the system starts with no customers. Let $L(t)$ be an integral valued non-increasing step function given by:

a) $L(0) = N$,

b) $L(t_i^*) = N - i$, $\qquad i = 1, 2, \dots, N$

where $t_N^* \leq T$.

Beérkezett: 1973. december 20.

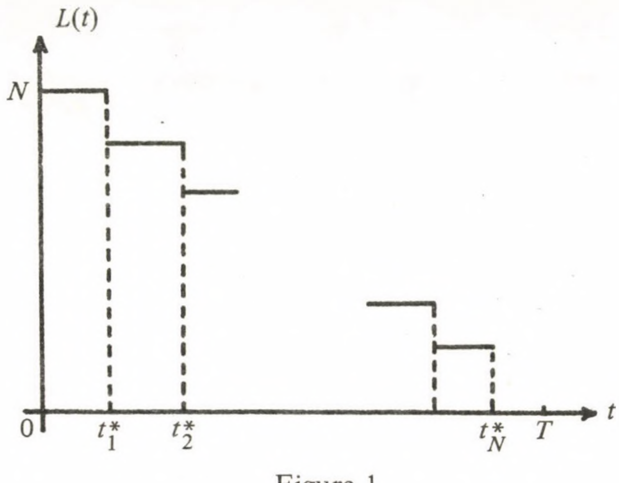

Figure 1.

In this section we show that the random variable τ given by $\tau = \inf_{0<u<T} \{u : Q(u) \geqslant L(u)\}$ is of mixed type i.e the distribution function of τ is a mixture of an atomic and continous distribution. An algorithm is presented to obtain a formula for that distribution function and its expectation and variance.

Define the event

$$E_t = \{Q(u) \geqslant L(u) \text{ for some } 0 \leqslant u \leqslant t\}, \quad \text{and}$$

(1)

$$B_k(t) = Pr\{\bar{E}_t \cap (Q(t) = k)\}, \qquad k = 0, 1, \ldots, L(t) - 1$$

where $\bar{E}_t = \{Q(u) < L(u) \text{ for all } 0 \leqslant u \leqslant t\}$, is the complement event of E_t.

Form (1) and Fig. 1, it can be easily seen that τ may assume the discrete values t_i^*'s with probabilities

(2) $$P(\tau = t_i^*) = B_{L(t_i^*-0)-1}(t_i^* - 0), \qquad i = 1, 2, \ldots, N;$$

the queue size process touches the function $L(t)$ at t_i^* from the left. Also τ may assume any value lies between the t_i^*'s since the queue size process may touch $L(t)$ from below as a result of an arrival. It follows that

(3) $$P(\tau \leqslant u) = \lambda \int_{R(u)} B_{L(t)-1}(t)dt + \sum_{t_i^* \leqslant u} B_{L(t_i^*-0)-1}(t_i^* - 0),$$

where $R(u) = \{t : t \in (0,u), t \neq t_i^*, \quad i = 1, \ldots, N\}$.

Our problem now is to give an explicit formula for the functions $B_k(t)$. We show that $B_k(t)$, $k = 0, 1, \ldots,$ satisfies different systems of linear differential equations on different intervals.

From equation (1) it follows that for small interval h

$$B_0(t+h) = B_0(t)(1-\lambda h) + \sigma(h) \qquad t^*_{N-1} \leqslant t < t^*_N$$

then $\quad B'_0(t) = -\lambda B_0(t)\,.$

Also $\quad B_0(t+h) = B_0(t)(1-\lambda h) + B_1(t)\mu h + \sigma(h)\,, \qquad 0 < t \leqslant t^*_{N-1}\,,$

then $B'_0(t) = -\lambda B_0(t) + \mu B_1(t).$

Similarly it can be shown that

$$B'_1(t) = -(\lambda+\mu)B_1(t) + \lambda B_0(t)\,, \qquad t^*_{N-2} < t < t^*_{N-1}\,,$$

$$B'_1(t) = -(\lambda+\mu)B_1(t) + \lambda B_0(t) + \mu B_2(t)\,, \qquad 0 < t \leqslant t^*_{N-2}\,,$$

$$B'_2(t) = -(\lambda+\mu)B_2(t) + \lambda B_1(t)\,, \qquad t^*_{N-3} < t \leqslant t^*_{N-2}\,,$$

$$B'_2(t) = -(\lambda+\mu)B_2(t) + \lambda B_1(t) + \mu B_3(t)\,, \qquad 0 < t \leqslant t^*_{N-3}\,,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$B'_{N-1}(t) = -(\lambda+\mu)B_{N-1}(t) + \lambda B_{N-2}(t)\,, \qquad 0 < t \leqslant t^*_1\,.$$

It is more suitable to rewrite these equations in the following form

if $\quad t^*_{N-1} < t \leqslant t^*_N\,, \qquad$ then

$$B'_0(t) = -\lambda B_0(t)\,,$$

if $\quad t^*_{N-2} < t \leqslant t^*_{N-1}\,, \qquad$ then

$$B'_0(t) = -\lambda B_0(t) + \mu B_1(t)\,,$$

$$B'_1(t) = -(\lambda+\mu)B_1(t) + \lambda B_0(t)\,,$$

if $\quad t^*_k < t \leqslant t^*_{k+1}\,, \qquad k = 0, 1, \ldots, N-2\,,$

$$B'_0(t) = -\lambda B_0(t) + \mu B_1(t)\,,$$

$$B'_1(t) = -(\lambda+\mu)B_1(t) + \lambda B_0(t) + \mu B_2(t)\,,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$B'_{N-(k+2)}(t) = -(\lambda+\mu)B_{N-(k+2)}(t) + \lambda B_{N-(k+3)}(t) + \mu B_{N-(k+1)}(t)\,,$$

$$B'_{N-(k+1)}(t) = -(\lambda+\mu)B_{N-(k+1)}(t) + \lambda B_{N-(k+2)}(t)\,.$$

Since the system starts with no customers at $t = 0$, then

$$B_0(0) = 1\,,$$

(4) $\qquad B_k(0) = 0\,, \qquad k = 1, 2, \ldots, N-1\,.$

We start by solving the last system of N linear differential equations on the interval $(0,t_1^*)$ using the initial values given by (4). Then we can determine the values of $B_0(t)$, $B_1(t), \ldots, B_{N-1}(t)$ at $t = t_1^*$ and use these values as the initial condition of the next system of $N-1$ linear differential equations on the interval (t_1^*, t_2^*). On repeating this procedure we can get the forms of $B_k(t)$ on different intervals.

THE SOLUTION OF THE FIRST SYSTEM OF LINEAR EQUATIONS

In matrix form this system can be written in the form

$$\frac{d}{dt} B(t) = AB(t) \tag{5}$$

where

$$B(t) = \begin{bmatrix} B_0(t) \\ B_1(t) \\ \cdot \\ \cdot \\ \cdot \\ B_{N-1}(t) \end{bmatrix} \qquad A = \begin{bmatrix} -\lambda & \mu & 0 \ldots & 0 \\ \lambda & -(\lambda+\mu) & \mu \ldots & 0 \\ 0 & & & \cdot \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \cdot & & & 0 \\ \cdot & & & \mu \\ 0 & \ldots & \ldots \lambda & -(\lambda+\mu) \end{bmatrix} \quad (N \times N)$$

with

$$B(0) = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \cdot \\ \cdot \\ 0 \end{bmatrix} \quad (N \times 1)$$

We seek the solution in the form

$$B(t) = h\, e^{wt}$$

where

$$h = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \cdot \\ \cdot \\ \cdot \\ \alpha_N \end{bmatrix} \quad (N \times 1)\,.$$

Then from equation (5), it follows that

(6) $$(A - wI)\,h = 0\,.$$

In order to obtain a non-trivial solution, it is necessary and sufficient that

(7) $$|A - wI| = 0\,.$$

Concerning the roots of this characteristic equation we prove.

Lemma 1. *All the roots of the characteristic equation (7) are negative and distincit.*

Proof. Denote

$$D_n(w) = \begin{vmatrix} \lambda + w & -\mu & 0 & \dots & 0 \\ -\lambda & \lambda+\mu+w & -\mu & \dots & 0 \\ 0 & & & & . \\ . & & & & . \\ . & & & & . \\ . & & & & . \\ . & & & & 0 \\ . & & & & -\mu \\ 0 & \dots & & -\lambda & \lambda+\mu+w \end{vmatrix} \qquad (n \times n)$$

It is easy to see that

(8) $$D_n(w) = (\lambda + \mu + w) D_{n-1}(w) - \lambda\mu D_{n-2}(w)\,, \qquad n \geqslant 2$$

where $D_0(w) = 1$, $D_1(w) = \lambda + w$ also $D_n(0) = \lambda^n > 0$ for all $n \geqslant 1$.
We prove now by using the recurrence relation (8) that all the roots of $D_N(w) = 0$ are distincit and negative. $D_1(w) = 0$ has only one root $w_1^{(1)} = -\lambda$
but $D_2(w) = (\lambda + \mu + w) D_1(w) - \lambda\mu$.
Since $D_2(0) = \lambda^2 > 0$, $D_2(-\lambda) < 0$ and $D_2(-\infty) > 0$ it follows that $D_2(w) = 0$ has two distincit negative roots $w_1^{(2)}$, $w_2^{(2)}$ such that

$$0 > w_1^{(2)} > -\lambda > w_2^{(2)}$$

generally if the roots of $D_{n-2}(w) = 0$ are

$$w_1^{(n-2)}, \quad w_2^{(n-2)}, \dots, w_{n-2}^{(n-2)}\,,$$

and the roots of $D_{n-1}(w) = 0$ are $w_1^{(n-1)}$, $w_2^{(n-1)}, \dots, w_{n-1}^{(n-1)}$ such that

$$0 > w_1^{(n-1)} > w_1^{(n-2)} > w_2^{(n-1)} > w_2^{(n-2)} >, \dots, > w_{n-2}^{(n-2)} > w_{n-1}^{(n-1)}$$

then $D_{n-2}(w_1^{(n-1)}) > 0$, $D_{n-2}(w_2^{(n-1)}) < 0$, $D_{n-2}(w_3^{(n-1)}) > 0, \dots$
i.e. sign $D_{n-2}(w_k^{(n-1)}) = (-1)^{k-1}$, $k = 1, 2, \dots, n-1$.
Then from the recurrence relation (8) it follows that the sign of $D_n(w_k^{(n-1)}) =$
$= -\lambda\mu D_{n-2}(w_k^{(n-1)})$ alternates i.e the roots of $D_n(w) = 0$ are separated by the roots of $D_{n-1}(w) = 0$.
Consequently all the roots are negative and distincit.

We get a system of N solutions

$$B_{(1)}(t) = h^{(1)} e^{w_1^{(N)} t}, \quad B_{(2)}(t) = h^{(2)} e^{w_2^{(N)} t}, \dots, B_{(N)}(t) = h^{(N)} e^{w_N^{(N)} t},$$

and the general solution of the system (5) is

$$\text{(9)} \qquad B(t) = \sum_{i=1}^{N} d_i B_{(i)}(t) = \sum_{i=1}^{N} d_i h^{(i)} e^{w_i^{(N)} t}$$

where d_i, $i = 1, 2, \dots, N$ are arbitrary constants to be determined from the initial condition (4) at $t = 0$.
The general solution (9) can be written in the form

$$B_0(t) = \sum_{i=1}^{N} d_i \alpha_1^{(i)} e^{w_i^{(N)} t},$$

$$\text{(9')} \qquad B_1(t) = \sum_{i=1}^{N} d_i \alpha_2^{(i)} e^{w_i^{(N)} t}$$

$$\dots\dots\dots\dots\dots$$

$$B_{N-1}(t) = \sum_{i=1}^{N} d_i \alpha_N^{(i)} e^{w_i^{(N)} t}.$$

The constants d_i's are determined from

$$1 = \sum_{i=1}^{N} d_i \alpha_1^{(i)},$$

$$0 = \sum_{i=1}^{N} d_i \alpha_2^{(i)},$$

$$\dots\dots\dots\dots$$

$$0 = \sum_{i=1}^{N} d_i \alpha_N^{(i)}.$$

The Column vector h^* associated with a root w^* according to the equation

$$\text{(10)} \qquad Ah^* = w^* h^*$$

is given by Lemma 2.

Lemma 2. *The componants of the eigen vector* h^* *corresponding to* w^* *are*

$$\alpha_k^* = \frac{D_{k-1}(w^*)}{\mu^{k-1}} \alpha_1^* , \qquad k = 2, \ldots, N$$

where $D_k(w)$ *is defined by the recurrence relation (8).*

Proof: The coefficient matrix of the equation (10) is

$$\begin{bmatrix} -(\lambda + w^*) & \mu & 0 \ldots & 0 \\ \lambda & -(\lambda + \mu + w^*) & \mu \ldots & 0 \\ 0 & & & \cdot \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \cdot & & & 0 \\ \cdot & & & \mu \\ 0 & & & \lambda - (\lambda + \mu + w^*) \end{bmatrix} \quad (N \times N)$$

since the determinant of this matrix is $D_N(w^*) = 0$ but $D_{N-1}(w^*) \neq 0$, it follows that the rank of this matrix is $N - 1$ and the first $N - 1$ rows are linearly independent while the last row is a linear combination of the others. It follows from the first equation in the system (10), that

$$\alpha_2^* = \frac{(\lambda + w^*)}{\mu} \alpha_1^* = \frac{D_1(w^*)}{\mu} \alpha_1^*$$

from the second equation

$$\alpha_3^* = \frac{(\lambda + \mu + w^*)}{\mu^2} \alpha_2^* - \frac{\lambda}{\mu} \alpha_1^* = \frac{(\lambda + \mu + w^*)D_1(w^*) - \lambda\mu}{\mu^2} \alpha_1^* = \frac{D_2(w^*)}{\mu^2} \alpha_1^*$$

similarly

$$\alpha_4^* = \frac{D_3(w^*)}{\mu^3} \alpha_1^* ,$$

$$\alpha_N = \frac{D_{N-1}(w^*)}{\mu^{N-1}} \alpha_1^* ,$$

where α_1^* can be arbitrarly chosen, say $\alpha_1^* = 1$.

Now the solution of the system (5) is well defined and the values of the functions $B_0(t), B_1(t), \ldots, B_{N-1}(t)$ can be obtained at the end t_1^* of the first interval.

Repeating the same steps for the smaller system of $N-1$ linear differential equations on the interval (t_1^*, t_2^*) and so on untill the last system, which contains one differential equation on the interval (t_{N-1}^*, t_N^*), is reached.
The distribution function of τ given by (3) can be written in the form

if $\quad 0 < u < t_1^*$

$$P(\tau \leqslant u) = \lambda \int_0^u B_{N-1}(t)dt = \lambda \sum_{i=1}^{N} \frac{d_i \alpha_N^{(i)}}{w_i^{(N)}} (e^{w_i^{(N)} u} - 1),$$

if $\quad t_1^* < u < t_2^*$

$$P(\tau \leqslant u) = \lambda \int_0^{t_1^*} B_{N-1}(t)dt + B_{N-1}(t_1^*) + \lambda \int_{t_1^*}^{u} B_{N-2}(t)dt = \tag{11}$$

$$= \lambda \sum_{i=1}^{N} \frac{d_i \alpha_N^{(i)}}{w_i^{(N)}} (e^{w_i^{(N)} t_i^*} - 1) + \sum_{i=1}^{N} d_i \alpha_N^{(i)} e^{w_i^{(N)} t_1^*} +$$

$$+ \lambda \sum_{i=1}^{N-1} \frac{\bar{d}_i \bar{\alpha}_{N-1}^{(i)}}{w_i^{(N-1)}} (e^{w_i^{(N-1)} u} - e^{w_i^{(N-1)} t_1^*})$$

where $\bar{d}_i$'s are the arbitrary constants on the interval (t_1^*, t_2^*), $\bar{\alpha}_{N-1}^{(i)}$ represents the componants of the eigen vectors corresponding to the eigen values on the interval (t_1^*, t_2^*).
$P(\tau \leqslant u)$ where $t_{k-1}^* < u < t_k^*$, $\quad k = 3, \dots, N$ can be easily obtained in the same way.

2. AN M/M/1 ASSOCIATED WITH REWARDS

We consider an M/M/1 which offers service for the arriving customers during a finite operating time $(0, T)$. Every served customer provides a revenue $r > 0$. At the closing time T no new customers are admitted and the present customers, if any, are to be served in an overtime, at a runing cost C/unit time with $r < c/\mu$. The system has to be operated in order to make the expected net revenue as high as possible. The influence of the overtime costs on the net revenue, necessitates choosing a policy to control the input process. Such a policy may take the form of a decreasing function $L(t)$ that give the upper limit of the number of customers present in the system at time t. Because of the steplikeness of the process $Q(t)$, one can reduce the function space $\{L(t), \quad 0 \leqslant t \leqslant T\}$ to the space of integer valued non-increasing step functions. According to such a policy the input is closed at the moment

$$\tau = \inf_{0 < u < T} \{u : Q(u) \geqslant L(u)\}$$

(i.e no customer is admitted after τ). Then a question arises how to choose $L(t)$ from the space of integer-valued non-increasing step functions in order to make the net revenue as high as possible.

THE OPTIMAL POLICY $L(t)$

Let $f(k, t)$ denotes the total expected overtime costs when there are k customers in the system at time t (t is measured in the negative direction where the origin represents the closing time T) ignoring later arrivals, then

$$(12) \qquad f(k,t) = c \int_t^\infty (x-t)\,\gamma_k(x)\,dx\,, \qquad k \geqslant 1$$

where $\gamma_i(x) = \mu^i x^{i-1} e^{-\mu x}/(i-1)! \qquad x \geqslant 0\,.$

At any moment $0 < u < T$, the net revenue of the queue operator according to a policy $\hat{L}(u)$ is $r\hat{L}(u) - f(\hat{L}(u), u)$.
The optimal policy $L(u)$ is that policy which maximizes the difference $r\hat{L}(u) - f(\hat{L}(u), u)$ for all $0 < u < T$. The points of jumps for the optimal $L(t)$ function can be determined as follows:

Define.

$$g_t(n) = rn - f(n,t)\,,$$

$$\Delta g_t(n) = g_t(n+1) - g_t(n) = r - \Delta f(n,t)$$

where

$$(13) \qquad \Delta f(n,t) = f(n+1,t) - f(n,t) = \frac{c}{\mu}\sum_{j=0}^{n}\frac{(\mu t)^j}{j!}\,e^{-\mu t}$$

since $\Delta f(n, t)$ is increasing in n and decreasing in t, then $\Delta g_t(n)$ is decreasing in n and increasing in t.
Thus $L(t)$ is the first integer for which $\Delta g_t(n) < 0$, i.e $L(t) = \min\{n \geqslant 0;\ g_t(n) < 0\}$.
At $t = 0$ we have

$$\Delta g_0(n) = r - \Delta f(n),\ 0 = r - c/\mu < 0$$

for any $n \geqslant 0$.

Therefore $L(0) = 0$. $L(t)$ remains zero untill the point $t_0 = \{t : \Delta g_t(0) = 0\}$ is reached, since $\Delta g_t(0)$ is increasing in t. Now $\Delta g_{t_0}(0) = 0$, but $\Delta g_{t_0}(1) < 0$ since $\Delta g_t(n)$ is decreasing in n. It follows that $L(t_0) = 1$.
Similarly $L(t) = 1$ for $t \geqslant t_0$ untill the point $t_1 = \{t : \Delta g_t(1) = 0\}$ is reached, at t_1, $L(t)$ must jump from 1 to 2 in-order to keep $\Delta g_t(.)$ negative.
With the same argument, we see that $L(t)$ jumps from i to $i+1$ at the point $t_i = \{t : \Delta g_t(i) = 0\}$. If the maximum value of $L(t)$ on the interval $(0, T)$ is N, then it is convenient to use the transform

$$t_i^* = T - t_{N-1}\,, \qquad i = 1, 2, \ldots, N\,,$$

so that all the times are measured from the opening time $t = 0$. The individual customers optimal balking strategy for the system M/M/1 was given in [1] and has a similar nature as $L(t)$.

3. THE EXPECTED NET REVENUE ASSOCIATED WITH $L(t)$

Let $\{N_t, t \geqslant 0\}$ be the poisson process denoting the number of the arrivals during $(0, t)$. It follows from [3] that for the stopping time τ

$$EN_\tau = \lambda E\tau .$$

If we denote

R_A : the reward associated with the admitted customers during $(0, \tau)$,

V: the overtime cost associated with the policy $L(t)$, then we have noting that our assumptions imply that $\tau > 0$,

$$E(R_A \mid \tau > 0) = \lambda r E N_\tau = \lambda r E\tau \tag{14}$$

$$E(V \mid \tau > 0) = cE \sup \left(\sum_{i=1}^{L(\tau)} \xi_i - (T - \tau), 0 \right) = \tag{15}$$

$$= c \int_0^{t_N^*} E\left(\left[\sum_{i=1}^{L(\tau)} \xi_i - (T - \tau) \right]^+ \mid \tau = t \right) dP_\tau(t)$$

where ξ_i's are independently and identically exponentially distributed random variables with mean $1/\mu$ which represent the service time, and

$$P_\tau(t) = P(\tau \leqslant t) .$$

The integral in (15) can be written as the sum of integrals according to the nature of the distribution function of τ.

Therefore from (2) and (9) we have

$$E(V \mid \tau > 0) = \lambda \sum_{j=1}^{N} \int_{t_{j-1}^*}^{t_j^*} f(N - j + 1, T - t) B_{N-j}(t) dt + \tag{15'}$$

$$+ \sum_{j=1}^{N} f(N - j, T - t_j^*) \cdot P(\tau = t_j^*) .$$

The expected net revenue given that $\tau > 0$ is

$$E(R_A \mid \tau > 0) - E(V \mid \tau > 0) . \tag{16}$$

We compare now the random stopping time policy and the deterministic optimal rejection time policy from the point of view of the expected net revenue associated with each. First we consider the queuing system M/M/1 for which a) $\rho = \lambda/\mu < 1$, b) the initial distribution of the queue size is stacionary one:

$$p_0^* = P\{Q(0) = 0\} = 1 - \rho ,$$

$$p_k^* = P\{Q(0) = k\} = (1 - \rho)\rho^k , \qquad k \geqslant 1 .$$

The random stopping time policy states that $\tau = 0$ whenever the system starts with N or more customers and $\tau > 0$ when the system starts with $N - 1$ or less customers. Then we get

$$P(\tau = 0) = \sum_{i=N}^{\infty} p_i^* = \rho^N ,$$

$$P(\tau > 0) = 1 - \rho^N .$$

It follows that the distribution function of τ will take the form

$$P(\tau \leqslant t) = P(\tau = 0) + P(\tau \leqslant t, \tau > 0) = \rho^N + (1 - \rho^N)P(\tau \leqslant t \mid \tau > 0) \tag{17}$$

where $P(\tau \leqslant t \mid \tau > 0)$ is given by (3) with the initial condition

$$B_k(0) = \frac{p_k^*}{\sum_{i=0}^{N-1} p_i^*} , \qquad k = 0, 1, \ldots , N - 1 .$$

If we denote

R_I : the reward associated with the initial number of customers,

R : the total reward, then $R = R_I + R_A$.

From the law of total expectation:

$$\begin{aligned} ER &= \rho^N E(R \mid \tau = 0) + (1 - \rho^N)E(R \mid \tau > 0) = \\ &= \rho^N \frac{r \sum_{k=N}^{\infty} kp_k^*}{\rho^N} + (1 - \rho^N)\left\{\frac{r \sum_{k=1}^{N-1} kp_k^*}{1 - \rho^N} + E(R_A \mid \tau > 0)\right\} = \\ &= \frac{\rho r}{1 - \rho} + (1 - \rho^N)\lambda r E\tau \end{aligned} \tag{18}$$

also by (12)

$$\begin{aligned} EV &= \rho^N E(V \mid \tau = 0) + (1 - \rho^N)E(V \mid \tau > 0) = \\ &= \sum_{k=N}^{\infty} p_k^* f(k, T) + (1 - \rho^N)E(V \mid \tau > 0) . \end{aligned} \tag{19}$$

Thus the expected net revenue associated with the random stopping time policy is

(20) $ER - EV$.

On the other hand when the case of the deterministic optimal rejection time policy is considered for the same system., from [2] we can deduce easily that the expected net revenue $C(t')$ associated with a rejection time t' is given by

$$C(t') = \left(\lambda t' + \frac{\rho}{1-\rho} \right) r + c\rho \frac{e^{(\lambda-\mu)(T-t')}}{\lambda-\mu}$$

taking the derivative with respect to t', and equating with zero, then

$$\frac{d}{dt'} c(t') = \lambda \left[r - \frac{c}{\mu} e^{(\lambda-\mu)(T-t')} \right] = 0 ,$$

or $$T - t' = \frac{1}{\lambda-\mu} \ln \left(\frac{r\mu}{c} \right)$$

(21) $$t' = T - \frac{1}{\lambda-\mu} \ln \left(\frac{r\mu}{c} \right) .$$

Noting that

$$\frac{d^2}{dt'^2} c(t') = \rho c (\lambda-\mu) e^{(\lambda-\mu)(T-t')} < 0$$

for all t' it follows that $C(t')$ attains its maximum at t' given by (21).

The optimal rejection time = $t^* = \max(0, t')$ i.e

(22) $$t^* = \max \left(0, T + \frac{1}{\mu-\lambda} \ln \left(\frac{r\mu}{c} \right) \right) .$$

The associated Expected net revenue = $C(t^*)$ =

(23) $$= \left(\lambda t^* + \frac{\rho}{1-\rho} \right) r + c\rho \frac{e^{(\lambda-\mu)(T-t^*)}}{\lambda-\mu} .$$

4. DESCRIPTION OF THE PROGRAM USED FOR THE COMPUTATIONS

A computer program is written to fulfil the calculations necessary for obtaining numerical results. Firstly we should give the parameters λ, μ, r, C, T . Then the control function $L(t)$ should be described. This is made by the means of the points $t_1^*, t_2^*, \ldots, t_N^*$. If one would like to use the optimal control policy then the t_i^*'s have to be determined in accordance with sec.2. To do this the subroutine DELFNT calculates the value of $\Delta f(n, t)$ for $n \geqslant 1, t > 0$. Then the subroutine POINT determines the values $t_i = \{ t : r = \Delta f(i, t) \}$ where the t_i's are measured from the closing time T in the negative direction.

After this we turn to find the roots of $D_n(w) = 0$, $n = 1, 2, \ldots, N$. This is done by the means of the recurrence relation (8) and the property of the sequence of roots proved. Lemma 2. is applied in calculating the eigen vectors corresponding to the intervals. After this it remains only to determine the arbitrary constants. Starting from the first interval we use the initial condition (4). The $B_k(t_1^*)$, $k = 0, 1, \ldots, N-1$ values serve for the initial condition on the next interval and so on.

These quantities are quite enough to determine the mean value $E\tau$ and the expected net revenue.

As a way of checking the results obtained, the system was simulated and all the quantities of interest were estimated and compared with the corresponding calculated quantities. The results obtained by both methods showed a good agreement. Point out the effectiveness of the exast procedure, we mention that for getting the numerical results presented in the foregoing tables 1 minute 47 seconds was needed for the exact values while for those of simulated 17 minutes, 19 seconds.

In order to make the comparison mentioned in sec.3, we should start the system from the stationary state. This is equivalent to use the initial condition

$$B_k(0) = p_k^* / \sum_{i=0}^{N-1} p_i^* , \qquad k = 0, 1, \ldots, N-1$$

for the first interval.

The expected net revenue associated with the deterministic optimal rejection time is easily obtained by (23). The results obtained given that the system has been started with a stationary queue size are summarized in the following tables:

For $r = 2$, $C = 50$,

	λ	μ	T	N	cal. $E\tau$	sim. $E\tau$	cal. EV	sim. EV
1	0.5	0.75	14	5	4.570	4.573	0.0947	0.0972
2	0.6	0.80	13	5	3.386	3.391	0.2919	0.3043
3	0.6	0.80	15	6	4.829	4.832	0.2025	0.2155
4	0.5	0.75	18	7	8.013	7.986	0.0469	0.0477

This table compare between the calculated and the simulated values for $E\tau$ and the expected overtime $E\,V$.

	λ	μ	T	N	E_D	cal. E_R	sim. E_R
1	0.5	0.75	14	5	– 0.0263	3.8320	3.719
2	0.6	0.80	13	5	– 7.9263	– 4.5331	– 5.119
3	0.6	0.80	15	6	– 3.3350	1.6702	1.093
4	0.5	0.75	18	7	3.9737	9.6650	9.588

where E_D : the expected net revenue associated with deterministic optimal rejection time policy.

E_R : the expected net revenue associated with the random stopping time policy.

From the second table it follows that the random stopping time policy seems to be more better than the deterministic one.

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Summary

An M/M/1 queueing system with a simple cost structure is considered, assuming that the system operates during a finite interval after which any remaining customers will require extra overtime service costs. For controlling the input a random rejection time, which is the first passage time that the queue size hints a given non-increasing positive $L(t)$ function, is discussed. Its distribution function is obtained by solving successive systems of differential equations. A computational procedure has been written and the numerical results obtained are presented showing that the effectiveness of the random rejection policy is higher than that of a deterministic one.

Резюме

Рассматривается система обслуживания M/M/m , функционирующая в конечном интервале времени. Требования, оставшиеся в системе по окончании периода функционирования, дообслуживаются сверхурочно, что приводит к дополнительной оплате. Чтобы уменьшить сверхурочную работу, рассматривается управление входящим потоком, по которому вход системы закрывается, когда длина очереди пересекает данную невозрастающую функцию $L(t)$. Такой момент закрытия входа представляет собой случайную величину, независящую от будущего. Для нахождения её функции распределения необходимо решить последовательность систем дифференциальных уравнений. Написана вычислительная процедура и приведены численные результаты, показывающие, что рассматренное правило закрытия входа более эффективно, чем правило детерминированного типа.

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